Calculation of an axial temperature distribution using the reflection coefficient of an acoustic wave

Milan Červenka and Michal BednaříkCC

Citation: The Journal of the Acoustical Society of America **138**, EL359 (2015); doi: 10.1121/1.4931830
View online: http://dx.doi.org/10.1121/1.4931830
View Table of Contents: http://asa.scitation.org/toc/jas/138/4
Published by the Acoustical Society of America

Calculation of an axial temperature distribution using the reflection coefficient of an acoustic wave

Milan Červenka and Michal Bednařík
Czech Technical University in Prague, Faculty of Electrical Engineering, Technická 2, 166 27 Prague 6, Czech Republic
milan.cervenka@fel.cvut.cz, bednarika@fel.cvut.cz

Abstract: This work verifies the idea that in principle it is possible to reconstruct axial temperature distribution of fluid employing reflection or transmission of acoustic waves. It is assumed that the fluid is dissipationless and its density and speed of sound vary along the wave propagation direction because of the fluid temperature distribution. A numerical algorithm is proposed allowing for calculation of the temperature distribution on the basis of known frequency characteristics of reflection coefficient modulus. Functionality of the algorithm is illustrated on a few examples, its properties are discussed.

© 2015 Acoustical Society of America
[CC]
Date Received: March 2, 2015 **Date Accepted:** September 10, 2015

1. Introduction

Interaction of thermal and acoustic fields plays an important role in many applications such as thermoacoustic devices, combustion, and power generation systems, etc. The measurement of a fluid temperature employing the temperature dependence of its speed of sound and the time that an acoustic impulse travels between a transmitter and receiver is a well established technique; utilizing several transmitters and receivers, the technique can also be extended for the temperature distribution measurement.[1]

Whilst some effort was made[2–5] to understand propagation of acoustic waves in ducts with axial temperature gradients, there is a question whether the fluid temperature distribution can be inferred from the properties of acoustic waves reflected from (or transmitted through) the temperature-inhomogeneous region, which is the subject of this letter.

Geometrical arrangement of the problem can be seen in Fig. 1. A harmonic wave travelling from region ($x \le 0$) with constant mean temperature T_{0a} partially reflects from and transmits through a temperature-inhomogeneous region with mean temperature distribution $T_{0b} = T_{0b}(x)$, $0 < x < L$, to a semi-infinite region ($x \ge L$) with constant mean temperature T_{0c}.

Section 2 of this article summarizes the theory of one-dimensional wave propagation in temperature-inhomogeneous fluids and gives an elementary example of reflection coefficient properties. Section 3 describes the algorithm proposed for the calculation of a temperature distribution from frequency characteristics of reflection coefficient, the results of which are presented in Sec. 4. Some concluding remarks are given in Sec. 5.

2. Theoretical background

2.1 Wave equation

One-dimensional wave equation for a temperature-inhomogeneous inviscid, non-thermal-conducting fluid follows from the linearised Euler's and continuity equations

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial x}, \quad \frac{\partial \rho'}{\partial t} = -\frac{\partial}{\partial x}(\rho_0 v), \tag{1}$$

where p' is the acoustic pressure, v is the acoustic velocity, ρ' is the acoustic density, x is the spatial coordinate, t is the time, and $\rho_0 = \rho_0(x)$ is the ambient fluid density. It is assumed that the fluid density ρ_0 varies in space because of the temperature distribution and that the ambient pressure p_0 is constant. Then if we assume $p = p(\rho, s)$, where s is the specific entropy, we can write[3,6]

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \left(\frac{\mathrm{D}p}{\mathrm{D}s}\right)_\rho \frac{\mathrm{D}s}{\mathrm{D}t} + \left(\frac{\mathrm{D}p}{\mathrm{D}\rho}\right)_s \frac{\mathrm{D}\rho}{\mathrm{D}t} = c_0^2 \frac{\mathrm{D}\rho}{\mathrm{D}t}, \quad \Rightarrow \quad \frac{\partial p'}{\partial t} = c_0^2\left(\frac{\partial \rho'}{\partial t} + v\frac{\partial \rho_0}{\partial x}\right), \tag{2}$$

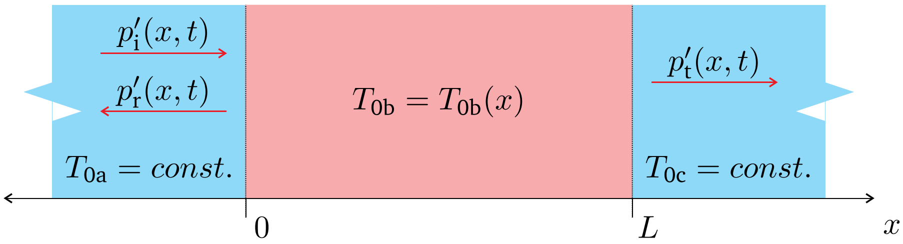

Fig. 1. (Color online) Geometry of the problem.

as it is assumed that the specific entropy of an acoustic particle is constant. Here, D represents the material derivative, $p = p_0 + p'$ is the total pressure, $\rho = \rho_0 + \rho'$ is the total density, $c_0 = c_0(x) = \sqrt{\gamma R T_0(x)}$ is the isentropic speed of sound, γ is the Poisson's constant, R is the specific gas constant, and $T_0 = T_0(x)$ is the ambient temperature. Only the linear terms are retained in Eq. (2).

The acoustic quantities are further considered to be time-harmonic, we can write $\zeta(x,t) = \Re[\hat{\zeta}(x) \exp \mathrm{i}\omega t]$, where $\hat{\zeta}$ stands for the phasor (complex amplitude) of the given quantity and ω is the angular frequency. Equations (1) and (2) can be combined into the Helmholtz-type equation[2–4]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\rho_0(x)}\frac{\mathrm{d}\hat{p}'}{\mathrm{d}x}\right] + \frac{\omega^2}{\rho_0(x)c_0^2(x)}\hat{p}' = 0, \tag{3}$$

where the state equation for an ideal gas $p_0 = \rho_0(x) R T_0(x)$ is employed for the calculation of $\rho_0(x)$. Equation (3) can be solved with the radiation boundary conditions

$$\frac{\mathrm{d}\hat{p}'}{\mathrm{d}x} - \mathrm{i}k_\mathrm{a}\hat{p}' = -2\mathrm{i}k_\mathrm{a}\hat{P}_{\mathrm{i}0} \quad \text{at} \quad x = 0 \quad \text{and} \quad \frac{\mathrm{d}\hat{p}'}{\mathrm{d}x} + \mathrm{i}k_\mathrm{c}\hat{p}' = 0 \quad \text{at} \quad x = L, \tag{4}$$

where $k_\mathrm{a} = \omega/c_0$ for $x \le 0$, $k_\mathrm{c} = \omega/c_0$ for $x \ge L$, and $\hat{P}_{\mathrm{i}0}$ is (arbitrary) complex amplitude of the incident wave $\hat{p}'_\mathrm{i} = \hat{P}_{\mathrm{i}0}\exp(-\mathrm{i}k_\mathrm{a}x)$. Because $\hat{p}' = \hat{p}'_\mathrm{i} + \hat{p}'_\mathrm{r}$ for $x \le 0$, the reflection coefficient modulus can be calculated from solution of Eq. (3) with the boundary conditions [Eq. (4)] as

$$|\mathcal{R}| = \left|\frac{\hat{p}'_\mathrm{r}(0)}{\hat{p}'_\mathrm{i}(0)}\right| = \left|\frac{\hat{p}'(0) - \hat{p}'_\mathrm{i}(0)}{\hat{p}'_\mathrm{i}(0)}\right|. \tag{5}$$

2.2 An elementary example: An uniform temperature distribution

Here it is assumed that $T_{0\mathrm{a}} = T_{0\mathrm{c}} \neq T_{0\mathrm{b}} = const.$, thus, $\rho_{0\mathrm{b}}$ and $c_{0\mathrm{b}}$ are constant in the interjacent region, and the solution of Eq. (3) can be found using elementary methods; employing the continuity of acoustic velocity and pressure at $x = 0$ and $x = L$ and the relation $Z_i = p_0\sqrt{\gamma/RT_{0i}}$, $i =$ a, b, c for the characteristic impedances yields in

$$|R| = \left|\frac{(1 - T_{0\mathrm{b}}/T_{0\mathrm{a}})(\mathrm{e}^{-2\mathrm{i}k_\mathrm{b}L} - 1)}{\left(1 - \sqrt{T_{0\mathrm{b}}/T_{0\mathrm{a}}}\right)^2 \mathrm{e}^{-2\mathrm{i}k_\mathrm{b}L} - \left(1 + \sqrt{T_{0\mathrm{b}}/T_{0\mathrm{a}}}\right)^2}\right|, \tag{6}$$

where $k_\mathrm{b} = \omega/c_{0\mathrm{b}}$. The reflection coefficient modulus [Eq. (6)] is a periodic function of frequency, see Fig. 2, it has zero values at frequencies

$$f_m = \frac{m c_{0\mathrm{b}}}{2L} = \frac{m}{2L}\sqrt{\gamma R T_{0\mathrm{b}}}, \quad m = 0, 1, 2, \ldots \tag{7}$$

and the maxima of

$$|\mathcal{R}|_{\max} = \left|\frac{T_{0\mathrm{a}} - T_{0\mathrm{b}}}{T_{0\mathrm{a}} + T_{0\mathrm{b}}}\right| \qquad \text{for} \qquad f_m = \frac{1 + 2m}{4L}\sqrt{\gamma R T_{0\mathrm{b}}}, \quad m = 0, 1, 2, \ldots\,. \tag{8}$$

This means that if the maximum of the reflection coefficient $|\mathcal{R}|_{\max}$ is known (e.g., measured) as well as the frequency difference Δf between two neighbouring zeros (maxima), the temperature $T_{0\mathrm{b}}$ and the length L can be calculated as

$$T_{0\mathrm{b}} = T_{0\mathrm{a}}\frac{1 \pm |\mathcal{R}|_{\max}}{1 \mp |\mathcal{R}|_{\max}}, \quad L = \frac{\sqrt{\gamma R T_{0\mathrm{b}}}}{2\Delta f}, \tag{9}$$

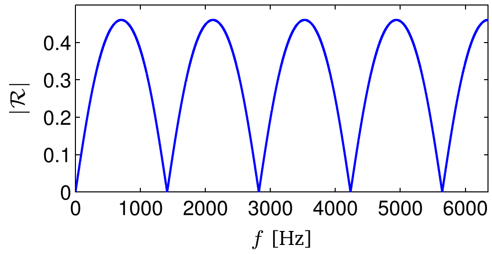

Fig. 2. (Color online) Reflection coefficient modulus [Eq. (6)] for air, $T_{0a} = T_{0c} = 293.15\,\text{K}$, $T_{0b} = 813.15\,\text{K}$, $L = 0.2\,\text{m}$.

where the upper sign applies if $T_{0b} > T_{0a}$, the lower then for $T_{0b} < T_{0a}$. Whether the $T_{0b} > T_{0a}$ or not is in many cases apparent from the physical context.

2.3 Riccati equation

In cases where an analytical solution of Eq. (3) is not known, it must be sought numerically; together with boundary conditions [Eq. (4)] employing, e.g., the shooting method,[7] which is based on repeated numerical integration, and it is thus numerically demanding.

Another, simpler approach is based on the Riccati equation[8] for specific acoustic impedance Z. Employing Eqs. (1) and (2), it can be shown that

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\hat{p}'}{\hat{v}}\right) = \frac{1}{\hat{v}}\frac{\mathrm{d}\hat{p}'}{\mathrm{d}x} - \frac{Z}{\hat{v}}\frac{\mathrm{d}\hat{v}}{\mathrm{d}x} = -\mathrm{i}\omega\rho_0(x) + \frac{\mathrm{i}\omega}{\rho_0(x)c_0^2(x)}Z^2. \tag{10}$$

The reflection coefficient for the wave impinging upon the temperature inhomogeneous region can be calculated as follows. For $x \geq L$, $T_0(x) = T_{0c}$. As the domain $x \geq L$ is homogeneous and infinite and there are no sources, there is only wave travelling in the positive direction, which means that the specific acoustic impedance at $x = L$ has the form $Z(L) = Z_c = \rho_{0c}c_{0c}$. This impedance is used as an initial condition for numerical integration of Eq. (10) backward from $x = L$ to $x = 0$ resulting in the knowledge of specific acoustic impedance $Z(0)$.

For $x \leq 0$ the temperature is constant $T_0(x) = T_{0a}$. There are two waves, one going rightward to the inhomogeneous region and the other, which is reflected from it. In this region, we can write for specific acoustic impedance

$$Z(x) = \frac{\hat{p}'_{\mathrm{i}}(x) + \hat{p}'_{\mathrm{r}}(x)}{\hat{v}_{\mathrm{i}}(x) + \hat{v}_{\mathrm{r}}(x)} = \rho_{0a}c_{0a}\frac{\hat{p}'_{\mathrm{i}}(x) + \hat{p}'_{\mathrm{r}}(x)}{\hat{p}'_{\mathrm{i}}(x) - \hat{p}'_{\mathrm{r}}(x)}, \tag{11}$$

as $\hat{p}'_{\mathrm{i}} = \rho_{0a}c_{0a}\hat{v}_{\mathrm{i}}$ and $\hat{p}'_{\mathrm{r}} = -\rho_{0a}c_{0a}\hat{v}_{\mathrm{r}}$. Substituting $x = 0$ into Eq. (11) results in

$$\mathcal{R} = \frac{\hat{p}'_{\mathrm{r}}(0)}{\hat{p}'_{\mathrm{i}}(0)} = \frac{Z(0) - \rho_{0a}c_{0a}}{Z(0) + \rho_{0a}c_{0a}}. \tag{12}$$

Calculation of the reflection coefficient using Eqs. (10) and (12) is numerically much more efficient than using Eq. (3) as this is an initial-value problem for a first-order differential equation.

3. The algorithm

The algorithm for reconstruction of the temperature distribution $T_{0b}(x)$ for $0 < x < L$ from the frequency characteristics of the reflection coefficient modulus works as follows. Let us assume that temperatures T_{0a} and T_{0c} are known and $T_{0a} = T_{0c}$ and $T_{0b}(x) \geq T_{0a}$ or $T_{0b}(x) \leq T_{0a}$ for $0 < x < L$. Let us assume further that the reflection coefficient modulus is known (for example, it is prescribed or has been measured) at some frequencies f_n so that we have $|\mathcal{R}_{\mathrm{pr}}(f_n)|$, $n = 1, 2, \ldots, N$. We can then parametrize the not-yet-known temperature $T_{0b}(x)$ using a parameter vector $\boldsymbol{q}$, so that

$$T_{0b}(x) = T_{0b}(x, \boldsymbol{q}), \quad \boldsymbol{q} = [q_1, q_2, \ldots, q_M]. \tag{13}$$

The reflection coefficient modulus calculated for this temperature distribution is then a function of the frequency and the parameter vector $\boldsymbol{q}$, let us denote it $|\mathcal{R}_{\mathrm{ca}}(f, \boldsymbol{q})|$.

The parameter vector $\boldsymbol{q}$ and thus the temperature distribution can be calculated by minimization of an M-dimensional objective function

$$\mathcal{Q}(\boldsymbol{q}) = \frac{1}{N}\sum_{n=1}^{N}\left[|\mathcal{R}_{\rm pr}(f_n)| - |\mathcal{R}_{\rm ca}(f_n,\boldsymbol{q})|\right]^2. \tag{14}$$

The parametrization [Eq. (13)] should be chosen in order that it could describe the reconstructed (expected) temperature-distribution well even with not-too-large number of parameters M as the computational effort needed for the global minimum finding in multidimensional space increases quickly with the problem dimensionality.

The number of frequencies N and the frequency range needed for the reconstruction depend on the details of the temperature distribution and the used parametrization. Generally, spatially small temperature variations require wider frequency range.

4. Numerical results

In this section, several numerical results are given that should validate the idea of reconstructing the temperature distribution from the frequency characteristics of the reflection coefficient modulus. Here the temperature was parametrized by means of the cubic splines using control points

$$[x_i, T_i] = [0, T_{0a}], [\xi_1 L, T_1], [\xi_2 L, T_2], \ldots, [\xi_K L, T_K], [L, T_{0c}], \tag{15}$$

where $0 < \xi_i < 1$ for $i = 1,\ldots, K$. These $2K+1$ parameters are supplemented with the derivatives of the temperature at the boundaries of the interval so that the parameter vector has the form

$$\boldsymbol{q} = \left[\xi_1, \ldots, \xi_K, T_1, \ldots, T_K, \left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{x=0+}, \left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{x=L-}, L\right] \tag{16}$$

and thus $M = 2K+3$.

Air was used as the medium in which the sound waves propagate, $T_{0a} = T_{0c} = 293.15\,\mathrm{K}$ ($= 20\,^\circ\mathrm{C}$). For calculation of the reflection coefficient $|\mathcal{R}_{\rm ca}(f,\boldsymbol{q})|$, the Riccati equation (10) and relation [Eq. (12)] were used, the objective function [Eq. (14)] for $M = 9$ was minimized using a self-adaptation variant (μ, λ)–ES of Evolution Strategies (Ref. 9) implemented in C++. To avoid any bias (in fact, the prescribed temperature distributions are known here), the individual components of the parameter vector [Eq. (16)] were in some reasonable range generated randomly. For example, one minimization of function [Eq. (14)] using $N = 100$ frequencies, population of $\lambda = 105$ individuals, evolution for 200 generations (which are the typical values used here), lasts 70 s of running time on a PC with CPU Intel Core i7–2600, 3.4 GHz. In all the cases, the algorithm was run several times with different (random) initial conditions in order to increase the probability that there is not a better solution than the one found and that the parameters are set appropriately.

Figure 3 shows the results of the numerical experiments. In all the cases, reflection coefficients moduli $|\mathcal{R}_{\rm pr}(f_n)|$ were calculated in COMSOL MULTIPHYSICS for frequencies $f_n = 50 \times n$ Hz, $n = 1, 2,\ldots,$ using Eq. (3) with radiation boundary conditions [Eq. (4)] for individual temperature distributions with $T_{\min} = 293.15\,\mathrm{K}$ ($=20\,^\circ\mathrm{C}$) and $T_{\max} = 813.15\,\mathrm{K}$ ($= 520\,^\circ\mathrm{C}$). Left panels in the figure show the comparison of the prescribed reflection coefficients $|\mathcal{R}_{\rm pr}(f)|$ and the corresponding coefficients $|\mathcal{R}_{\rm ca}(f)|$ for the parameter vector $\boldsymbol{q}$ minimizing the objective function [Eq. (14)]. The right panels in the figure show the comparison of the prescribed temperature distributions (using which $|\mathcal{R}_{\rm pr}(f)|$ were calculated) with the calculated (reconstructed) temperature distributions. The diamond marks show the positions of the control points [Eq. (15)].

It can be seen that the prescribed and reconstructed temperature distributions correspond to each other very well. The only exception is the "triangular" temperature distribution, which is plausible as the cubic splines are smooth functions. In case of non-smooth distributions, it would be straightforward to replace the cubic splines, e.g., by piecewise linear interpolation.

5. Discussion and conclusions

It has been shown that in principle, it is possible to reconstruct the temperature distribution of a fluid region employing the knowledge of reflection coefficient modulus frequency characteristics; similarly, the transmission coefficient modulus information could be used as well. The proposed technique could be utilized, e.g., for non-invasive flame temperature distribution measurement or in applications that are based on inherent interactions of acoustic and thermal fields. The method, similarly as it was shown in Sec. 2.2, cannot discriminate whether the temperature in the inhomogeneous region is higher or lower than in the surrounding regions, which, however, can be in many

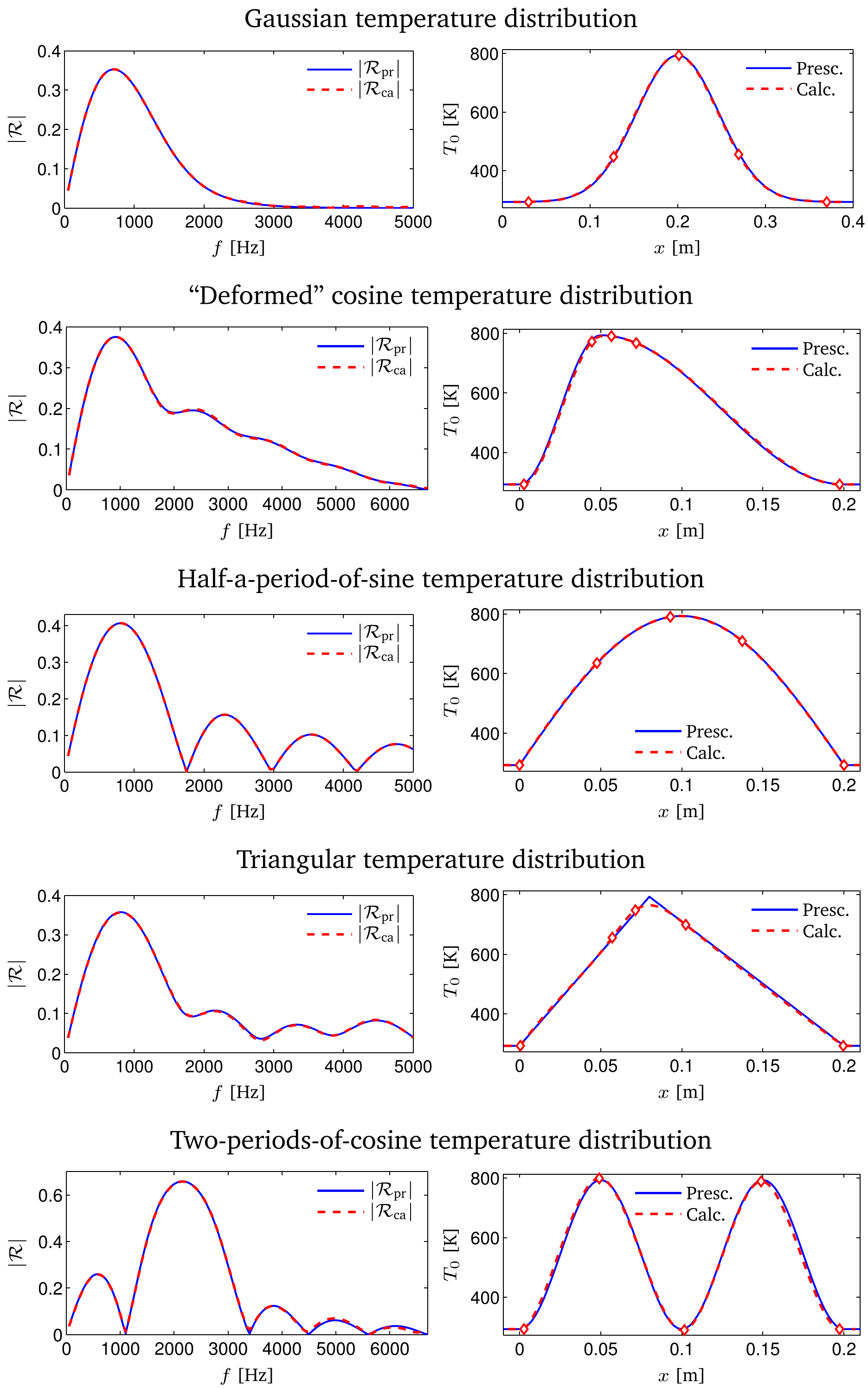

Fig. 3. (Color online) Comparison of prescribed and reconstructed reflection coefficient moduli and corresponding temperature distributions.

cases deduced from the physical context. Similarly, the reflection coefficient modulus does not keep information about the absolute position of the inhomogeneous region or its orientation if it is non-symmetric relative to its centre.

For successful reconstruction of the temperature field, appropriate parametrization is essential, preferably based on some *a priori* information. Here cubic splines were employed; piecewise-linear interpolation of control points is another option for which the Riccati equation (10) can be integrated analytically.

Acknowledgments

This work was supported by GACR Grant No. 15-23079S.

References and links

[1]S. F. Green, "An acoustic technique for rapid temperature distribution measurement," J. Acoust. Soc. Am. **77**(2), 759–763 (1985).

[2]A. Kapur, A. Cummings, and P. Mungur, "Sound propagation in a combustion can with axial temperature and density gradients," J. Sound Vib. **25**(1), 129–138 (1972).

[3]A. Cummings, "Ducts with axial temperature gradients: An approximate solution for sound transmission and generation," J. Sound Vib. **51**(1), 55–67 (1977).

[4]R. I. Sujith, G. A. Waldherr, and B. T. Zinn, "An exact solution for one-dimensional acoustic fields in ducts with an axial temperature gradient," J. Sound Vib. **184**(3), 389–402 (1995).

[5]B. M. Kumar and R. I. Sujith, "Exact solution for one-dimensional acoustic fields in ducts with a quadratic mean temperature profile," J. Acoust. Soc. Am. **101**(6), 3798–3799 (1997).

[6]D. T. Blackstock, *Fundamentals of Physical Acoustics* (Wiley-Interscience, New York, 2000), Chap. 8, pp. 278–281.

[7]W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd ed. (Cambridge University Press, Cambridge, 2007), Chap. 18, pp. 959–961.

[8]K. Walther, "Reflection factor of gradual-transition absorbers for electromagnetic and acoustic waves," IEEE Trans. Antenn. Propag. **8**(6), 608–621 (1960).

[9]A. E. Eiben and J. E. Smith, *Introduction to Evolutionary Computing* (Springer, Berlin, 2010), Chap. 4, pp. 71–87.