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Citation: The Journal of the Acoustical Society of America **135**, EL134 (2014); doi: 10.1121/1.4865262
View online: http://dx.doi.org/10.1121/1.4865262
View Table of Contents: http://asa.scitation.org/toc/jas/135/3
Published by the Acoustical Society of America

Equations for description of nonlinear standing waves in constant-cross-sectioned resonators

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Abstract: This work is focused on investigation of applicability of two widely used model equations for description of nonlinear standing waves in constant-cross-sectioned resonators. The investigation is based on the comparison of numerical solutions of these model equations with solutions of more accurate model equations whose validity has been verified experimentally in a number of published papers.

PACS numbers: 43.25.Gf [MH]
Date Received: September 13, 2013 **Date Accepted:** January 19, 2014

1. Introduction

The description of the behavior of nonlinear standing waves in constant-cross-sectioned resonators represents an interesting problem both from a theoretical and practical point of view. There are a number of model equations which can be used for this purpose but limits of their applicability are not obvious.

This study aims to assess the validity of the Kuznetsov and inhomogeneous Burgers model equations [see, e.g., Kuznetsov (1970), Gusev (1984), Rudenko *et al.* (2001), Bednarik and Konicek (2004)] that include only nonlinear terms of second order. These equations are widely used for analysis of nonlinear processes in constant-cross-sectioned resonators, in particular in cylindrical ones. Our approach is based on the comparison of the above-mentioned simpler model equations with a more accurate equation containing the third-order terms [see Ilinskii *et al.* (1998)] whose validity has been verified experimentally, see, e.g., Lawrenson *et al.* (1998).

2. Model equation

A one-dimensional wave equation describing the high-amplitude acoustic field in a constant-cross-sectioned waveguide can be written, see Ilinskii *et al.* (1998), in the form

$$c_0^2\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial t^2}+\frac{b}{\rho_0}\frac{\partial^3\varphi}{\partial x^2\partial t}=2\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial t}+(\gamma-1)\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial x^2}+\frac{(\gamma+1)}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2}, \quad (1)$$

where φ is the velocity potential, $v=\partial\varphi/\partial x$ is the acoustic velocity, x is the spatial coordinate along the waveguide axis, t is the time, c_0 is the small-signal speed of sound, ρ_0 is the ambient fluid density, γ is the Poisson's exponent, and $b=\zeta+4\eta/3$ is an attenuation constant, where ζ and η are the bulk and shear viscosities. The first two terms on the left-hand side of Eq. (1) represent the linear lossless wave equation and the third term on the left-hand side describes the viscous volume attenuation. The first two terms on the right-hand side are quadratic-nonlinear ones and the last one is a cubic-nonlinear term. If $b=0$, Eq. (1) represents an exact wave equation for plane-waves in perfect gas, see, e.g., Hamilton and Morfey (1998). Acoustic pressure can be calculated from the solution of Eq. (1) as

$$p'=\frac{\rho_0c_0^2}{\gamma}\left\{1-\frac{\gamma-1}{c_0^2}\left[\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2-\frac{b}{\rho_0}\frac{\partial^2\varphi}{\partial x^2}\right]\right\}^{\gamma/(\gamma-1)}-\frac{\rho_0c_0^2}{\gamma}, \quad (2)$$

see Ilinskii *et al.* (1998). If the cubic-nonlinear term in Eq. (1) is dropped and $\partial^2\varphi/\partial t^2 = c_0^2\partial^2\varphi/\partial x^2$ is substituted into the corresponding terms (the third one on the left-hand side and the second one on the right-hand side), the error is a third-order small and the resulting equation reads

$$c_0^2\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial t^2} = \frac{\partial}{\partial t}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\gamma-1}{2c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{b}{\rho_0 c_0^2}\frac{\partial^2\varphi}{\partial t^2}\right], \tag{3}$$

which is the well-known Kuznetsov equation,[1] see Kuznetsov (1970).

Acoustic pressure can be calculated from the solution of Eq. (3) using Eq. (2) or, for consistency, it can be recast in the second approximation using the Taylor series expansion into the form

$$p' = -\rho_0\frac{\partial\varphi}{\partial t} - \frac{\rho_0}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\rho_0}{2c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + b\frac{\partial^2\varphi}{\partial x^2}. \tag{4}$$

If Eq. (1) or (3) is used for calculation of acoustic field in an acoustic resonator driven by a vibrating piston, they can be supplemented with the following boundary conditions:

$$\left(\frac{\partial\varphi}{\partial x}\right)_{x=0} = 0, \quad \left(\frac{\partial\varphi}{\partial x}\right)_{x=L} = v_p(t), \tag{5}$$

where L is the length of the resonant cavity and $v_p(t)$ is the velocity of the vibrating piston.

The inhomogeneous Burgers equation is another model equation which can be used for the description of nonlinear standing waves in the cylindrical resonator. The inhomogeneous Burgers equation can be derived from the Kuznetsov equation (3). There are several ways how to derive this model equation [see, e.g., Gusev (1984), Rudenko *et al.* (2001), Enflo and Hedgerg (2002)]. However, all of these derivation-procedures follow from the assumption that the acoustic field inside the cylindrical resonator can be described as a superposition of two nonlinear counter-propagating waves $\varphi_\pm$ which are coupled only by conditions on the resonator side-walls. This assumption means that we can ignore a mutual interaction of the counter-propagating waves. To justify the assumption we can use the method which has been sketched, e.g., in Gusev (2005) and Gusev *et al.* (1999).

Within the framework of the second order approximation of the nonlinear acoustics we can suppose that each of the counter-propagating waves is relatively weak in its amplitude and therefore the nonlinear processes are also weak. The relative weakness of the nonlinear processes allows us to suppose that wave profiles of the counter-propagating waves vary slowly with time and propagation distance, i.e., the changes in the wave profiles are assumed to be small for distances of order $O(\lambda)$ and times of order $O(T)$, where λ is the wavelength and T is the wave period. We can formalize the mentioned statement by the fact that we will search for the solution of the Kuznetsov equation (3) in the following form:

$$\varphi = \mu\varphi_+(t_1 = \mu t, x_1 = \mu x, \tau_+ = t - x/c_0) - \mu\varphi_-(t_1 = \mu t, x_1 = \mu x, \tau_- = t + x/c_0), \tag{6}$$

where $\mu < 1$ is a small scaling parameter and t_1 and x_1 represent slow time and space coordinates. After substituting the expression (6) into Eq. (3), considering $b \sim \mu$ and ignoring the terms of the order 3 and higher we obtain

$$\mu\frac{\partial^2\varphi_\pm}{\partial\tau_\pm\partial t_1} \pm c_0\mu\frac{\partial^2\varphi_\pm}{\partial\tau_\pm\partial x_1} \pm \frac{\beta}{2c_0^2}\left[\frac{\partial}{\partial\tau_\pm}\left(\frac{\partial\varphi_\pm}{\partial\tau_\pm}\right)^2 - \frac{\partial\varphi_+}{\partial\tau_+}\frac{\partial\varphi_-}{\partial\tau_-}\right] - \frac{b}{2\rho_0 c_0^2}\frac{\partial^3\varphi_\pm}{\partial\tau_\pm^3} = 0, \tag{7}$$

where $\beta=(\gamma+1)/2$. The second term in square brackets in Eq. (7) represents the interaction of the counterpropagating waves.

Equation (7) formally contains two unknowns ($\varphi_{\pm}$), two fast variables ($\tau_{\pm}$), and two slow variables (t_1, x_1). The fast variables are not independent because they are connected by the relation $\tau_- = \tau_+ + 2x/c_0$. Therefore in the coordinate system (t_1, x_1, τ_+), accompanying the wave propagating to the right, the function $\varphi_-(t_1, x_1, \tau_-) = \varphi_-(t_1, x_1, \tau_+ + 2x/c_0)$ is a fast varying function of coordinate x. After using the following operator:

$$\langle \cdots \rangle \equiv \frac{1}{\lambda}\int_{x-\lambda/2}^{x+\lambda/2} (\cdots)\mathrm{d}x, \tag{8}$$

we can conduct the formal procedure of the separation of the counter-propagating waves because φ_+ does not depend on the fast coordinate x and consequently the operator (8) does not influence φ_+ and its derivatives. Assuming the absence of an average directional motion of fluid and the wave periodicity we can write

$$\langle \varphi_- \rangle \equiv \frac{1}{\lambda}\int_{x-\lambda/2}^{x+\lambda/2} \varphi_-\left(t_1, x_1, \tau_+ + \frac{2x'}{c_0}\right)\mathrm{d}x' = \frac{1}{T}\int_{t-T/2}^{t+T/2} \varphi_-(t_1, x_1, \tau_+ + 2t')\,\mathrm{d}t' = 0. \tag{9}$$

Therefore by the averaging we obtain

$$\langle \varphi_- \rangle = 0, \quad \left\langle \frac{\partial \varphi_-}{\partial \tau_-} \right\rangle = 0, \quad \left\langle \frac{\partial}{\partial \tau_-}\left(\frac{\partial \varphi_-}{\partial \tau_-}\right)^2 \right\rangle = 0.$$

After application of the operator (8) in the coordinate system (t_1, x_1, τ_+) for Eq. (7) we obtain an equation only for φ_+. Repeating this procedure for the coordinate system (t_1, x_1, τ_-) accompanying the wave propagating to the left, we get an equation only for φ_-.

The separated equations have the form

$$\mu\frac{\partial^2 \varphi_\pm}{\partial \tau_\pm \partial t_1} \pm c_0\mu\frac{\partial^2 \varphi_\pm}{\partial \tau_\pm \partial x_1} \pm \frac{\beta}{2c_0^2}\frac{\partial}{\partial \tau_\pm}\left(\frac{\partial \varphi_\pm}{\partial \tau_\pm}\right)^2 - \frac{b}{2\rho_0 c_0^2}\frac{\partial^3 \varphi_\pm}{\partial \tau_\pm^3} = 0. \tag{10}$$

Within the framework of second order theory it is possible to consider the interactions of the counter-propagating waves to be ineffective, which means that effects of the interactions do not accumulate. Consequently, the operator (8) retains in Eq. (7) only the terms which provide the influence accumulating with propagation distance [Gusev *et al.* (1999)]. This conclusion has been clarified in the works Rudenko *et al.* (2001) and Rudenko (2009) whose explanation is based on an approximate solution of the one-dimensional Westervelt equation for ideal fluids. Taking into account the relation

$$v_\pm = \frac{\partial \varphi_\pm}{\partial x} = \mp\frac{1}{c_0}\frac{\partial \varphi_\pm}{\partial \tau_\pm} + \mu\frac{\partial \varphi_\pm}{\partial x_1} \tag{11}$$

and keeping the assumed second order of accuracy of Eq. (10) we can rewrite it for acoustic velocities as

$$\frac{\partial v_\pm}{\partial t} \pm c_0\frac{\partial v_\pm}{\partial x} - \frac{\beta}{c_0}v_\pm\frac{\partial v_\pm}{\partial \tau_\pm} - \frac{b}{2\rho_0 c_0^2}\frac{\partial^2 v_\pm}{\partial \tau_\pm^2} = 0. \tag{12}$$

Here Eq. (12) is expressed in the original physical coordinates t and x.

As the nonlinear standing wave in a resonator is given as the superposition of two nonlinear counter-propagating waves we can write

$$v = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi_+}{\partial x} - \frac{\partial \varphi_-}{\partial x} = v_+ - v_-. \tag{13}$$

The boundary conditions (5) can be formulated in this case as

$$v = (v_+ - v_-)_{x=0} = 0, \tag{14}$$

$$v = (v_+ - v_-)_{x=L} = v_p(t) = -v_0 \sin(\omega t), \tag{15}$$

and an initial condition can be given as

$$v_\pm(t=0) = 0, \tag{16}$$

where v_0 is the velocity amplitude of the exciting piston which vibrates with an angular frequency ω. The angular frequency is assumed to be equal to one of the resonator eigenfrequencies (the resonance condition) that is given as

$$\omega_n = \frac{n\pi c_0}{L}, \quad n = 1, 2.3, \ldots . \tag{17}$$

With respect to the conditions (14)–(16) it is possible to apply the method of successive-approximations to Eqs. (12) which leads to the following forced model equations [see, e.g., Gusev (1984)]:

$$\frac{\partial \bar{v}_\pm}{\partial t} - \frac{\beta}{c_0} \bar{v}_\pm \frac{\partial \bar{v}_\pm}{\partial \tau_\pm} - \frac{b}{2\rho_0 c_0^2} \frac{\partial^2 \bar{v}_\pm}{\partial {\tau_\pm}^2} = \frac{v_0 c_0}{2L} \sin(\omega \tau_\pm), \tag{18}$$

where $\bar{v}_\pm$ represents the acoustic velocity of the counter-propagating waves in the first approximation, see, e.g., Gusev (1984). Equation (18) represents the inhomogeneous Burgers equation.

The acoustic velocity of the standing wave can be calculated from solution of Eq. (18) [see Bednarik and Konicek (2004)] as

$$v(t,x) = \bar{v}_+(t,\tau_+) - \bar{v}_-(t,\tau_-) - \frac{v_0 x}{L} \cos\left(\frac{\omega x}{c_0}\right) \sin(\omega \tau). \tag{19}$$

3. Comparison of numerical solutions of individual model equations

The model equations (1) and (3) were solved numerically in the time domain using the algorithm proposed in Cervenka (2007) whereas Eq. (18) was solved numerically in the frequency domain, see, e.g., Ginsberg and Hamilton (1998). For all the numerical

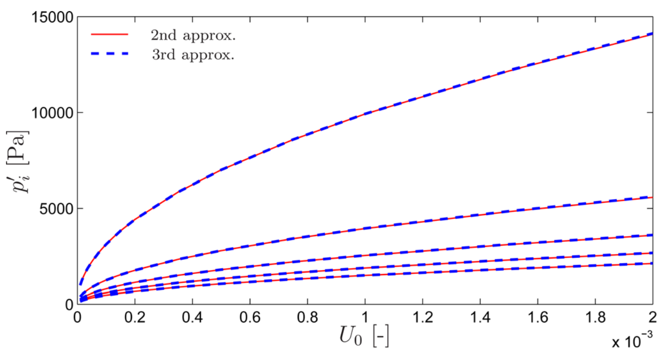

Fig. 1. (Color online) Comparison of amplitudes of the first five harmonics (one-side spectrum) of acoustic pressure at $x = 0$ calculated using Eqs. (1) and (3).

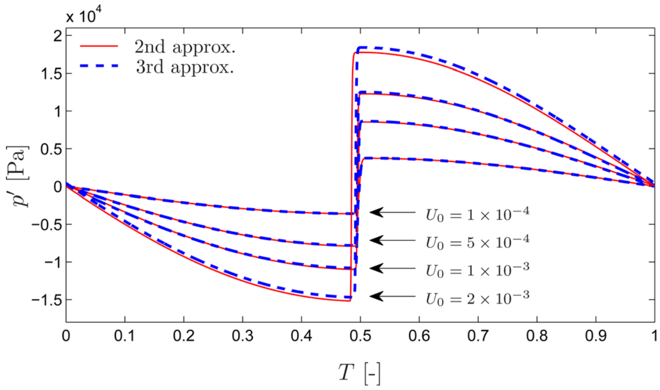

Fig. 2. (Color online) One period of steady-state acoustic pressure at $x = 0$ calculated using Eqs. (1) and (3).

calculations the driving piston vibrated at the fundamental eigen-frequency of ($\omega = \omega_1$) and air at room conditions was assumed as fluid and $G = \pi\omega b/(\rho_0 c_0^2) = 10^{-3}$. Numerical results are compared in Figs. 1–3. From these figures it is evident that there are only slight differences between the results of numerical solutions of the assumed model equations, even for relatively strong excitation of acoustic waves.

The comparison of numerical solutions of Eqs. (1) and (3) for the acoustic pressure is depicted in Figs. 1 and 2. It is obvious from Fig. 1 that the amplitudes of the first five harmonics are practically identical. One can observe a small discrepancy between the numerical solutions for $U_0 = v_0/(\pi c_0) = 2 \times 10^{-3}$ in Fig. 2 which is caused by the fact that a bit higher value of a dc-pressure follows from the model equation (1). The numerical solution for the acoustic velocity of all the model equations are sketched in Fig. 3. Again we can observe a slight difference of the solution of Eq. (1) with respect to the other solutions. This slight difference is reflected in a small shape asymmetry of the standing wave and is caused by a fine nonlinear resonance frequency shift following from the model equation (1). The numerical results show that unlike Eqs. (1) and (3), the inhomogeneous Burgers equation (18) does not capture this effect, however, relatively high nonlinear attenuation causes that the influence of the nonlinear resonance frequency shift can be ignored for acoustic fields up to the acoustic Mach number of about 0.1. The numerical solutions of Eqs. (3) and (18) are almost identical for all the driving amplitudes excluding the Gibbs oscillations.

4. Conclusion

In this work we presented three widely used model equations for the description of nonlinear standing waves in constant-cross-sectioned resonators with rigid ends. One

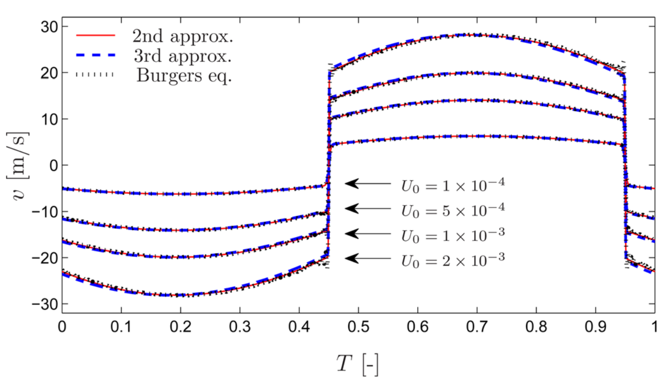

Fig. 3. (Color online) One period of steady-state acoustic velocity at $x = L/2$ calculated using Eqs. (1), (3), and (18).

of the model equations is more accurate by taking into account nonlinear effects of the third order. The remaining two model equations (3) and (18) can be derived from this more accurate one under certain assumptions. The model equations were solved numerically and numerical results were compared to each other to check whether acoustic fields inside the constant-cross-sectioned resonators satisfy the conditions under which the simpler model equations have been derived. By comparing the results of the model equations it can be stated that it is possible to use both the Kuznetsov and inhomogeneous Burgers equations for the description of nonlinear standing waves in the constant-cross-sectioned resonators even for relatively high sound pressure levels of practical interest (the acoustic Mach number of about 0.1). Unlike Eqs. (1) and (3), the inhomogeneous Burgers equation (18) does not capture the resonance frequency shift. If generation of higher harmonics is not suppressed, relatively high nonlinear attenuation prevents the effect of nonlinear resonance frequency shift to manifest itself in a substantial way. In the case of any method used for suppression of the higher harmonics it would probably be necessary to take into account this effect and to verify the applicability of the inhomogeneous Burgers equation using a more complex model.

From this conclusion it follows that in the case of relatively high nonlinear attenuation, an acoustic field inside the constant-cross-sectioned resonators can be described as a superposition of two nonlinear counter-propagating waves. It is also possible to ignore their mutual interactions because these waves couple only weakly and their coupling does not have cumulative character. For this reason we can use the far simpler inhomogeneous Burgers equation in comparison with Eq. (1) for analysis of nonlinear acoustic fields and a possible derivation of approximate analytical solutions.

Acknowledgments

This work was supported by GACR Grant No. P101/12/1925. The numerical calculations were performed at the Computing and Information Centre of CTU.

References and links

[1]The original derivation of this equation Kuznetsov (1970) also takes into account the influence of losses due to thermal conductivity which are incorporated into the parameter *b*.

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