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Michal Bednarik and Milan CervenkaOAS

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Finite amplitude standing waves in resonators terminated by a general impedance

Michal Bednarik[a] and Milan Cervenka
Department of Physics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technicka 2, 166 27 Prague, Czech Republic

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A general inhomogeneous Burgers equation describing finite-amplitude standing waves in resonators terminated by a general impedance is derived. This model equation can be used for modeling of nonlinear acoustic processes connected with some methods for enhancement of Q-factor of acoustic resonators. One of them is the method based on using a second-harmonics absorber. For better understanding of this method, it is convenient to know at least an approximate analytical solution of the model equation. This work presents some approximate solutions, which improve and extend the solutions that have been published previously. The solutions are compared with results obtained by numerical integration of the corresponding equations. © *2015 Acoustical Society of America*.
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I. INTRODUCTION

During the last two decades, we have been able to observe an increasing interest in the problems of nonlinear standing waves in fluid-filled resonators. This interest is caused by possibilities of using nonlinear acoustic waves for various practical purposes. There are a lot of methods which enable us to control frequency spectrum of the nonlinear standing waves. Most of the methods are focused on suppressing generation of higher harmonics for the purpose of acoustic energy accumulation in the first harmonic component. This method of control enables us to decrease an influence of nonlinear attenuation and to achieve very high amplitudes of nonlinear standing waves in the resonators, which leads to an enhancement of their Q-factor. The method called resonant macrosonic synthesis[1,2] (RMS) belongs among them. The RMS method uses the fact that the eigen-frequencies of the resonators are not equidistant; it means that higher eigen-frequencies are not integer multiples of the fundamental one. In these resonators, higher harmonics of the nonlinear standing wave do not coincide with higher eigen-frequencies. As a result, the higher harmonics are suppressed, thus, a shock front formation is prevented. Another method is based on two-frequency excitation of nonlinear acoustic field in a cylindrical resonator; see, e.g., Refs. 3 and 4. This method utilizes a parametric excitation of the first harmonics through the second one. The parametric excitation causes energy flow from the second harmonics, particularly, toward the first one. Amount and direction of the energy flow is controlled by both the phase difference between driving frequencies and the amplitude ratio of velocities of a driving piston. Another method uses a selective absorber in the cylindrical resonator.[5] The absorber attenuates the second harmonics, which develops in the course of cascade processes as a result of nonlinear interactions. The total suppression of the second harmonics brings about an interruption of the cascade processes and the acoustic energy is accumulated in the first harmonics. The method which uses a geometric nonlinearity caused by a boundary mobility[6] also belongs to the methods suppressing the generation of higher harmonics. In Ref. 6, it is shown that a boundary nonlinearity leads to a distortion of the temporal profile of the standing wave and to the generation of higher harmonics in the process of the development of steady-state oscillations. If a suitable displacement function of the driving piston is chosen, it is possible to enhance the Q-factor of the cylindrical resonator. Another method is described in Ref. 7. This method is based on an implementation of one of the acoustic resonator boundaries having a frequency-dependent impedance. It means that each reflection from this boundary introduces phase shifts between different harmonics destroying the shock front.

It is possible to combine some of the above-mentioned methods or supplement them with dispersion effects or multi-frequency excitation. Understanding nonlinear wave processes and assessment of their impact represents quite a complex problem. For this reason, it is useful to know approximate solutions of the model equations describing the mentioned methods. In Ref. 5, the approximate solution of a modified inhomogeneous Burgers equation, which models the method based on the selective absorption, was presented. However, this approximate solution is rather inaccurate and, in addition, it is limited only to the case of perfect fluids.

In this work, we have derived a relatively general model equation which enables us to describe finite-amplitude standing waves in a cylindrical resonator for three of the mentioned methods that suppress the higher harmonics generation. The derivation is presented in Sec. II. One advantage of Burgers-like equations is the possibility of finding their approximate solutions for configurations which are interesting both from theoretical and practical points of view. In Sec. III, we present new approximate solutions of the modified inhomogeneous Burgers equation modeling finite-amplitude standing waves in a resonator terminated by a

[a]Author to whom correspondence should be addressed. Electronic mail: bednarik@fel.cvut.cz

second-harmonics absorber and filled with an inviscid non-heat-conducting (perfect) fluid. In Sec. IV, these approximate solutions are generalized for thermo-viscous fluids. The penultimate section is devoted to comparing the analytical solutions with numerical ones and achieved results are summarized in Sec. V.

II. DERIVATION OF THE MODEL EQUATION

Using the basic equations of fluid-mechanics, namely, the Navier-Stokes equation of motion, the heat transfer equation, the equation of state,[8,9] and the continuity equation comprising the boundary layer effects,[10,11] we can derive in the second approximation the following one-dimensional modified Kuznetsov's equation,[11,12] for the velocity potential φ,

$$\frac{\partial^2\varphi}{\partial t^2}-c_0^2\frac{\partial^2\varphi}{\partial x^2}=\frac{b}{\rho_0 c_0^2}\frac{\partial^3\varphi}{\partial t^3}-\frac{\partial}{\partial t}\left[\frac{\beta-1}{c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\varphi}{\partial x}\right)^2\right]$$
$$+2B\frac{\partial^{3/2}\varphi}{\partial t^{3/2}}=0, \qquad (1)$$

where x is the spatial coordinate along the resonator axis, t is the time, c_0 is the small-signal sound speed, ρ_0 is the ambient density of the fluid, $b=\zeta+4\eta/3+\varkappa(1/c_V-1/c_p)$ is the coefficient of sound diffusivity, $\varkappa$ is the coefficient of heat conductivity, ζ and η are the coefficients of bulk and shear viscosity, β is the coefficient of nonlinearity, B is the boundary-layer parameter

$$B=\frac{\sqrt{\nu}}{r_0}\left(1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right), \qquad (2)$$

where ν is the kinematic viscosity, r_0 is the resonator inner radius, and $\mathrm{Pr}=\rho_0\nu c_p/\varkappa$ is the Prandtl number, $\gamma=c_p/c_V$ is the ratio of specific heats (c_p and c_V are the specific heats under constant pressure and volume, respectively). In the following text, we consider that $b\omega/(\rho_0 c_0^2)\sim B/\sqrt{\omega}\sim|\partial\varphi/\partial x|/c_0\sim\mu$, where $\mu\ll 1$ is a small scaling parameter, ω is a characteristic angular frequency, and $v=\partial\varphi/\partial x$ is an acoustic velocity.

In the second-order nonlinear theory, it is possible to describe the acoustic field in resonators as a superposition of two counter-propagating plane nonlinear waves and to neglect their mutual interaction in the case of not-too-strong fields.[13] The counter-propagating waves are coupled only by conditions on the side-walls of the resonators. In the frame of the second approximation, we can also neglect the fact that the exciting side-wall (piston) moves. Using the method of multiple scales and the above-mentioned assumptions, we can derive from the Kuznetsov's equation (1) the following model equations,[4,13–15] for the counter-propagating waves

$$c_0\frac{\partial v_+}{\partial x}+\frac{\partial v_+}{\partial t}-\frac{\beta}{c_0}v_+\frac{\partial v_+}{\partial\tau_+}=\frac{b}{2\rho_0 c_0^2}\frac{\partial^2 v_+}{\partial{\tau_+}^2}-B\frac{\partial^{1/2}v_+}{\partial\tau_+^{1/2}}, \qquad (3)$$

$$-c_0\frac{\partial v_-}{\partial x}+\frac{\partial v_-}{\partial t}-\frac{\beta}{c_0}v_-\frac{\partial v_-}{\partial\tau_-}=\frac{b}{2\rho_0 c_0^2}\frac{\partial^2 v_-}{\partial{\tau_-}^2}-B\frac{\partial^{1/2}v_-}{\partial\tau_-^{1/2}}, \qquad (4)$$

where $\tau_\pm=t\mp x/c_0$ is the retarded time, and $v_\pm$ is an acoustic velocity of the traveling waves, the sign "±" is connected with the traveling wave which propagates in the positive/negative direction of the x-axis.

All terms in Eqs. (3) and (4) are of the second order ($\sim\mu^2$), thus, x and t have to represent "slow" variables, whereas $\tau_\pm=t\mp x/c_0$ are "fast" variables, i.e., the retarded times contain, on the contrary, "fast" variables x and t.

On the basis of the solutions of Eqs. (3) and (4), we can write an acoustic velocity of the standing wave as

$$v=v_+-v_-. \qquad (5)$$

The angular eigen-frequencies (resonant angular frequencies) of resonators with rigid end caps filled with a perfect fluid are given by the following expression:

$$\omega_n=\frac{n\pi c_0}{L}=n\omega_1, \quad n=1,\,2,\,3,\ldots, \qquad (6)$$

where L is the length of the resonator.

If we consider the harmonic excitation of the standing waves with a piston at the position $x=0$ (see Fig. 1), we can express the boundary conditions as follows:

$$v=(v_+-v_-)_{x=0}=f(\omega t), \quad |f(\omega t)|/c_0\sim\mu^2, \qquad (7)$$

$$v=(v_+-v_-)_{x=L}=\int_{-\infty}^{\infty}\hat{\mathrm{T}}\mathrm{r}(\omega)\hat{v}_+(\omega)\exp(-\mathrm{j}\omega\tau_+)\mathrm{d}\omega, \qquad (8)$$

and we assume that

$$\left|\frac{1}{c_0}\int_{-\infty}^{\infty}\hat{\mathrm{T}}\mathrm{r}(\omega)\hat{v}_+(\omega)\exp(-\mathrm{j}\omega\tau_+)\mathrm{d}\omega\right|\sim\mu^2, \qquad (9)$$

where f is otherwise an arbitrary function with the period $T=2\pi/\omega$ representing a velocity of boundary (piston) vibrations, $\hat{\mathrm{T}}\mathrm{r}(\omega)$ is a frequency-dependent transmission coefficient

$$\hat{\mathrm{T}}\mathrm{r}(\omega)=\frac{2\rho_0 c_0}{\rho_0 c_0+\hat{Z}(\omega)}, \qquad (10)$$

where $\hat{Z}(\omega)$ represents a general impedance, and $\hat{v}_+(\omega)$ is a Fourier spectrum of the incident wave at $x=L$,

$$\hat{v}_+(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}v_+(t,\,\tau_+)\exp(\mathrm{j}\omega\tau_+)\mathrm{d}\tau_+. \qquad (11)$$

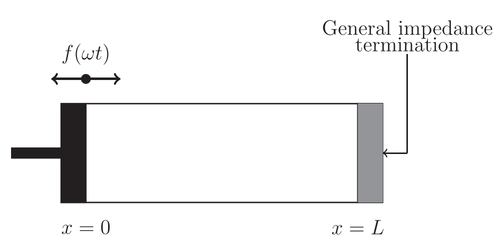

FIG. 1. Resonator with a general impedance.

We can seek a solution of the system of Eqs. (3) and (4) with the boundary conditions (7) and (8) and initial conditions $v_+(t=0) = v_-(t=0) = 0$ by means of the successive-approximation method.[16]

We assume that

$$v_\pm = v'_\pm + v''_\pm + \cdots, \quad |v'_\pm| \gg |v''_\pm| \cdots. \tag{12}$$

If we take into account only the first two terms of the series (12) and the relation (5), we can write

$$v = (v'_+ - v'_-) + (v''_+ - v''_-) + \mathcal{O}(\mu^3). \tag{13}$$

As the piston velocity amplitude is of the second order, then the boundary condition in the first approximation is equal to zero at $x = 0$, i.e.,

$$(v'_+ - v'_-)_{x=0} = 0. \tag{14}$$

The first-approximation equations have the following form:

$$\frac{\partial v'_\pm}{\partial x} = 0. \tag{15}$$

The solution of Eqs. (15) can be written as

$$v'_\pm = \bar{v}(t, \tau_\pm), \tag{16}$$

where $\bar{v}$ is an unknown function that must be determined using the boundary condition at $x = L$ [Eq. (8)]; see the following text.

It is evident that the solution (16) satisfies the condition (14).

From Eqs. (3) and (4), we can write the second-approximation equations

$$\frac{\partial v''_\pm}{\partial x} = \mp \frac{1}{c_0}\frac{\partial \bar{v}(t, \tau_\pm)}{\partial t} \pm \frac{\beta}{c_0^2}\bar{v}(t, \tau_\pm)\frac{\partial \bar{v}(t, \tau_\pm)}{\partial \tau_\pm} \pm \frac{b}{2\rho_0 c_0^3}\frac{\partial^2 \bar{v}(t, \tau_\pm)}{\partial {\tau_\pm}^2} \mp \frac{B}{c_0}\frac{\partial^{1/2}\bar{v}_\pm}{\partial \tau_\pm^{1/2}}. \tag{17}$$

Integrating Eq. (17) we obtain

$$v''_\pm = \left(\mp \frac{1}{c_0}\frac{\partial \bar{v}(t, \tau_\pm)}{\partial t} \pm \frac{\beta}{c_0^2}\bar{v}(t, \tau_\pm)\frac{\partial \bar{v}(t, \tau_\pm)}{\partial \tau_\pm} \pm \frac{b}{2\rho_0 c_0^3}\frac{\partial^2 \bar{v}(t, \tau_\pm)}{\partial {\tau_\pm}^2} \mp \frac{B}{c_0}\frac{\partial^{1/2}\bar{v}_\pm}{\partial \tau_\pm^{1/2}} \right) x + C_\pm, \tag{18}$$

where $C_\pm$ are integration "constants" which have to be determined on the basis of the boundary condition (7), i.e.,

$$C_\pm = \pm \frac{f(\omega \tau_\pm(x=0))}{2} = \pm \frac{f(\omega t)}{2}. \tag{19}$$

Considering Eqs. (18) and (19), we can write

$$v''(x=0) = (v''_+ - v''_-)_{x=0} = C_+ - C_- = f(\omega \tau_\pm(x=0)) = f(\omega t). \tag{20}$$

Employing the equality (13) and the condition (9), we can rewrite the boundary condition (8) into the following form:

$$v(x=L) = (v'_+ - v'_-)_{x=L} + (v''_+ - v''_-)_{x=L} = \int_{-\infty}^{\infty} \hat{\mathrm{T}}\mathrm{r}(\omega)\hat{\bar{v}}(\omega)\exp(-\mathrm{j}\omega\tau)\mathrm{d}\omega, \tag{21}$$

where

$$\hat{v}'_+(\omega) \equiv \hat{\bar{v}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{v}(t, \tau)\exp(\mathrm{j}\omega\tau)\mathrm{d}\tau \tag{22}$$

and

$$\tau \equiv \tau_+(x=L) = t - L/c_0, \quad v'_+(x=L) = \bar{v}(t, \tau). \tag{23}$$

Requiring satisfaction of the condition at $x = L$, we can obtain an equation for the quasi-periodic function, $\bar{v}$.

When we assume the perfectly rigid wall ($\hat{\mathrm{T}}\mathrm{r} = 0$) and that the frequency, ω, of vibrations of the boundary at $x = 0$ differs slightly from the eigen-frequency, ω_n, given by Eq. (6) ($\omega \approx \omega_n$), then we can write

$$\begin{aligned}
\bar{v}\big(t, \tau_-(x=L)\big) &= \bar{v}\left(t, \tau_+(x=L) + \frac{2L}{c_0}\right) \\
&= \bar{v}\left(t, \tau_+(x=L) + \frac{2\pi n}{\omega} + \frac{2L}{c_0} - \frac{2\pi n}{\omega}\right) \\
&= \bar{v}\left(t, \tau_+(x=L) + \frac{2\pi n}{\omega} + \frac{2\pi n}{\omega_n} - \frac{2\pi n}{\omega}\right) \\
&= \bar{v}\left(t, \tau_+(x=L) + \frac{2\pi n}{\omega} + \frac{2\pi n}{\omega}\delta\right) \\
&\simeq \bar{v}\left(t, \tau_+(x=L) + \frac{2\pi n}{\omega} + \frac{2\pi n}{\omega_n}\delta\right) \\
&= \bar{v}\left(t, \tau_+(x=L) + \frac{2\pi n}{\omega} + \frac{2L}{c_0}\delta\right) \\
&= \bar{v}\left(t, \tau_+(x=L) + \frac{2L}{c_0}\delta\right),
\end{aligned} \tag{24}$$

where $2L/c_0 = 2\pi n/\omega_n$, $2\pi n/\omega = nT$, and

$$\delta = \frac{\omega - \omega_n}{\omega_n}, \quad |\delta| \sim \mu. \tag{25}$$

On the basis of the Taylor series expansion for $x = L$, we can obtain

$$\bar{v}\big(t, \tau_-(x=L)\big) = \bar{v}\big(t, \tau_+(x=L)\big) + \frac{2L\delta}{c_0}\frac{\partial \bar{v}(t, \tau_+)}{\partial \tau_+}\bigg|_{x=L} + \mathcal{O}(\mu^3). \tag{26}$$

Substituting into the relation (21) from expressions (18) and (26) and denoting $\tau_-(x=L) \equiv \bar{\tau}$, we obtain

$$v(x=L) = -\frac{2L\delta}{c_0}\frac{\partial \bar{v}(t,\tau)}{\partial \tau} + \left(-\frac{1}{c_0}\frac{\partial \bar{v}(t,\tau)}{\partial t} + \frac{\beta}{c_0^2}\bar{v}(t,\tau)\frac{\partial \bar{v}(t,\tau)}{\partial \tau} + \frac{b}{2\rho_0 c_0^3}\frac{\partial^2 \bar{v}(t,\tau)}{\partial {\tau_+}^2} - \frac{B}{c_0}\frac{\partial^{\frac{1}{2}}\bar{v}(t,\tau)}{\partial \tau^{\frac{1}{2}}}\right)L$$
$$+\left(-\frac{1}{c_0}\frac{\partial \bar{v}(t,\bar{\tau})}{\partial t} + \frac{\beta}{c_0^2}\bar{v}(t,\bar{\tau})\frac{\partial \bar{v}(t,\bar{\tau})}{\partial \bar{\tau}} + \frac{b}{2\rho_0 c_0^3}\frac{\partial^2 \bar{v}(t,\bar{\tau})}{\partial \bar{\tau}^2} - \frac{B}{c_0}\frac{\partial^{\frac{1}{2}}\bar{v}(t,\bar{\tau})}{\partial \bar{\tau}^{\frac{1}{2}}}\right)L + f(\omega t)$$
$$= \int_{-\infty}^{\infty} \mathrm{Tr}(\omega)\hat{\bar{v}}(\omega)\exp(-\mathrm{j}\omega\tau)\mathrm{d}\omega. \tag{27}$$

From the relation (26) we can write

$$\bar{v}(t,\tau) = \bar{v}(t,\bar{\tau}) + \mathcal{O}(\mu^2). \tag{28}$$

Considering all terms between brackets in the relation (27) are of the second order ($\sim\mu^2$), then on the basis of the relation (28), we can replace all functions $\bar{v}(t,\bar{\tau})$ by the functions $\bar{v}(t,\tau)$. This substitution does not change the order of the supposed approximation. After this manipulation, we obtain the following equation:

$$\frac{\partial \bar{v}(t,\tau)}{\partial t} + \delta\frac{\partial \bar{v}(t,\tau)}{\partial \tau} - \frac{\beta}{c_0}\bar{v}(t,\tau)\frac{\partial \bar{v}(t,\tau)}{\partial \tau} - \frac{b}{2\rho_0 c_0^2}\frac{\partial^2 \bar{v}(t,\tau)}{\partial \tau^2} + B\frac{\partial^{1/2}\bar{v}(t,\tau)}{\partial \tau^{1/2}} + \frac{c_0}{2L}\int_{-\infty}^{\infty}\hat{\mathrm{T}}\mathrm{r}(\omega)\hat{\bar{v}}(\omega)\exp(-\mathrm{j}\omega\tau)\mathrm{d}\omega = \frac{c_0}{2L}f\left[\omega\left(\tau - \frac{L}{c_0}\right)\right]. \tag{29}$$

This model equation represents the generalized inhomogeneous Burgers equation.

We can modify the last term on the left-hand side of Eq. (29) by substitution from Eq. (22),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{v}(t,\tau')\hat{T}r(\omega)\exp[-\mathrm{j}\omega(\tau-\tau')]\mathrm{d}\omega\mathrm{d}\tau' = \int_{-\infty}^{\infty}\bar{v}(t,\tau')\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{T}r(\omega)\exp[-\mathrm{j}\omega(\tau-\tau')]\mathrm{d}\omega\right\}\mathrm{d}\tau' = \int_{-\infty}^{\infty}\bar{v}(t,\tau')\mathrm{Tr}(\tau-\tau')\mathrm{d}\tau', \tag{30}$$

where $\mathrm{Tr}(\tau)$ represents the kernel function.

On the basis of this result, it is possible to rewrite Eq. (29),

$$\frac{\partial \bar{v}(t,\tau)}{\partial t} + \delta\frac{\partial \bar{v}(t,\tau)}{\partial \tau} + \frac{c_0}{2L}\int_{-\infty}^{\infty}\bar{v}(t,\tau')\mathrm{Tr}(\tau-\tau')\mathrm{d}\tau' - \frac{\beta}{c_0}\bar{v}(t,\tau)\frac{\partial \bar{v}(t,\tau)}{\partial \tau} - \frac{b}{2\rho_0 c_0^2}\frac{\partial^2 \bar{v}(t,\tau)}{\partial \tau^2} + B\frac{\partial^{1/2}\bar{v}(t,\tau)}{\partial \tau^{1/2}} = \frac{c_0}{2L}f\left[\omega\left(\tau - \frac{L}{c_0}\right)\right]. \tag{31}$$

For better clarity, it is convenient to express this equation in the dimensionless form

$$\frac{\partial \bar{V}}{\partial \sigma} + \Delta\frac{\partial \bar{V}}{\partial \theta} + K\int_{-\infty}^{\infty}\bar{V}(\sigma,\theta')\mathrm{Tr}(\theta-\theta')\mathrm{d}\theta' - \bar{V}\frac{\partial \bar{V}}{\partial \theta} - \frac{1}{\Gamma}\frac{\partial^2 \bar{V}}{\partial \theta^2} + M\frac{\partial^{1/2}\bar{V}}{\partial \theta^{1/2}} = F(\theta - \pi\Omega), \tag{32}$$

where

$$\sigma = \frac{t}{t_s}, \quad \bar{V} = \frac{\bar{v}}{v_0}, \quad \theta = \omega\tau, \quad \Gamma = \frac{2\rho_0 c_0 \beta v_0}{b\omega},$$
$$v_0 = \sqrt{\frac{v_m c_0}{2\pi\beta}}, \quad M = \frac{c_0 B}{\beta v_0\sqrt{\omega}}, \quad t_s = \frac{c_0}{\beta\omega v_0},$$
$$K = \frac{c_0\Omega}{2\pi\beta v_0}, \quad \Delta = \frac{\delta c_0}{\beta v_0}, \quad F = \frac{f}{\Omega v_m}, \quad \Omega = \frac{\omega}{\omega_1}, \tag{33}$$

where v_m is a characteristic velocity.

Equation (31) enables us to model a relatively broad range of cases which are interesting from both the theoretical and practical points of view. Although this equation is relatively complex, its advantage is that it can be easily solved numerically and it is possible to derive a number of its approximate analytical solutions for the analysis of nonlinear acoustic processes.

Solving Eq. (31), we obtain a function, $\bar{v}(t,\tau)$. If we replace τ by $\tau_\pm$ in this function, we get functions $v'_\pm = \bar{v}(t,\tau_\pm)$ which allow us to write the solution for a standing wave in the first approximation as $v'(x,t) = \bar{v}(t,\tau_+) - \bar{v}(t,\tau_-)$.

III. RESONATOR TERMINATED BY THE SECOND-HARMONICS ABSORBER

Let us assume an ideal case when the general impedance termination represents a selective absorber which partly transmits the frequency, 2ω, and perfectly reflects all other frequencies from the resonator end. In this case, the selective absorber makes the second-harmonics generation weak and, therefore, reduces the cascade of the nonlinear transfer of energy upward over the spectrum.[5] Assuming $f(\omega(\tau - L/c_0)) = v_m\sin(\omega\tau)$ and the second-harmonics absorber, then we can replace the last term on the left-hand side of Eq. (29),

$$\frac{c_0}{2L}\int_{-\infty}^{\infty}\hat{\mathrm{T}}\mathrm{r}(\omega)\hat{\bar{v}}(\omega)\exp(-\mathrm{j}\omega\tau)\mathrm{d}\omega = \frac{c_0}{2L}\sum_{n=1}^{\infty}\alpha'_n b_n(t)\sin(n\omega\tau), \tag{34}$$

where $\omega = \omega_1$,

$$\alpha'_n = 0 \quad \text{for} \quad n \neq 2 \quad \text{and} \quad 0 \leq \alpha'_2 \leq 1. \tag{35}$$

When nonlinear wave processes are dominant and we can assume a perfect fluid, then we can rewrite Eq. (31) in the following form (see Ref. 5):

$$\frac{\partial \bar{v}(t,\tau)}{\partial t} - \frac{\beta}{c_0}\bar{v}(t,\tau)\frac{\partial \bar{v}(t,\tau)}{\partial \tau} = \frac{c_0 v_m}{2L}\sin(\omega\tau) - \alpha b_2(t)\sin(2\omega\tau), \tag{36}$$

where $\alpha = c_0\alpha'_2/(2L)$ is a selective-absorption coefficient and $b_2(t)$ is the amplitude of the second harmonics

$$b_2(t) = \frac{2}{\pi}\int_0^{\pi} \bar{v}(t,\tau)\sin(2\omega\tau)\mathrm{d}(\omega\tau). \tag{37}$$

Equation (36) can be also used for the second-harmonics absorber, which is realized by introducing resonant scatterers to the medium[5] (e.g., gas bubbles to a liquid). In this case, the above presented restriction for α'_2 does not apply.

We can express this equation in the dimensionless form

$$\frac{\partial \bar{V}}{\partial \sigma} - \bar{V}\frac{\partial \bar{V}}{\partial \theta} - \frac{1}{\Gamma}\frac{\partial^2 \bar{V}}{\partial \theta^2} = \sin\theta - DB_2\sin 2\theta, \tag{38}$$

where

$$D = \frac{\alpha c_0}{\beta\omega v_0}, \quad B_2 = \frac{2}{\pi}\int_0^{\pi} \bar{V}(\sigma,\theta)\sin 2\theta\,\mathrm{d}\theta. \tag{39}$$

On the basis of numerical solutions of Eq. (38), we can see in Fig. 2 that the second-harmonics absorber makes the second-harmonics generation weak with dependency on the parameter, D. This results in reducing the cascade processes, which causes an increase of the first-harmonics amplitude.

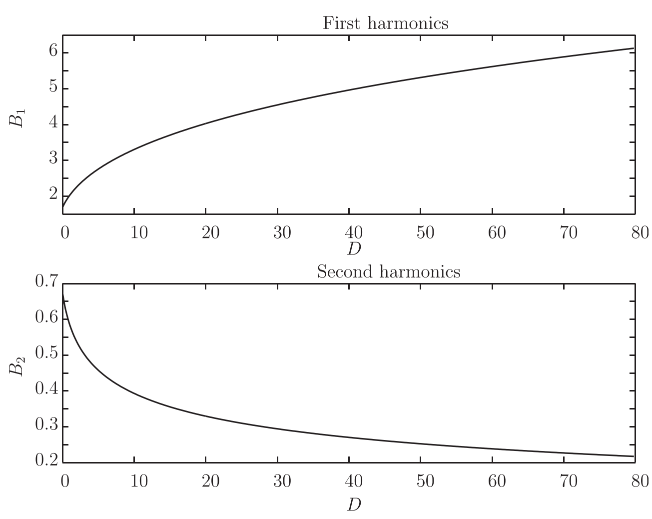

FIG. 2. Dependency of the first and second-harmonics amplitude on the parameter, D, for the steady state [numerically calculated using Eq. (37)].

A. Approximate solutions of the model equation (38)

Assuming nonlinear effects to be dominant (i.e., $\Gamma \gg 1$) and the steady state $[\partial\bar{V}(\sigma,\theta)/\partial\sigma = 0]$, then it is possible to consider the fluid to be perfect and we can rewrite Eq. (38) into the form

$$-\frac{1}{2}\frac{\mathrm{d}V^2}{\mathrm{d}\theta} = \sin\theta - DB_2\sin 2\theta, \tag{40}$$

where $\bar{V}(\sigma,\theta) = V(\theta)$.

Integration of Eq. (40) results in the following solution:

$$V(\theta) = \pm\sqrt{q' + 2\cos\theta - DB_2\cos 2\theta}, \tag{41}$$

where q' is an integration constant.

The relation (41) can be modified as follows:

$$V(\theta) = \pm\sqrt{q + 4\cos^2\frac{\theta}{2} + 8DB_2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}, \tag{42}$$

where $q \geq 0$ is a constant.

As the steady state solution is periodic, it obeys the conditions[5]

$$V(\theta = -\pi) = V(\theta = \pi) = 0. \tag{43}$$

To accomplish the conditions (43), it is necessary to put $q = 0$.

The mean value of V is assumed to be zero

$$\langle V\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\theta)\mathrm{d}\theta = 0. \tag{44}$$

The condition (44) is satisfied when the shock position $\theta_s = 0$ for the time interval $\langle -\pi, \pi\rangle$. In this case, the solution (42) can be modified into the form

$$V(\theta) = 2\cos\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}}\,\mathrm{sign}(\theta), \tag{45}$$

where $\mathrm{sign}(\theta)$ is the signum function.

The position of the maximum value of the function (45) is for $DB_2 > 0.5$ at

$$\theta_{\max} = 2\arccos\sqrt{\frac{1 + DB_2}{4DB_2}}. \tag{46}$$

The maximum value of the function (45) is given as

$$V_{\max} \equiv V(\theta_{\max}) = \frac{1 + 2DB_2}{\sqrt{2DB_2}}. \tag{47}$$

The position of the maximum value $\theta_{\max}$ is approaching $\pi/2$ with increasing value of DB_2.

To complete the solution (45), it is necessary to find the second-harmonics amplitude B_2,

$$B_2 = \frac{2}{\pi}\int_0^{\pi} V(\theta)\sin 2\theta\,\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\pi} 2\cos\frac{\theta}{2}\sqrt{1+2DB_2\sin^2\frac{\theta}{2}}\sin 2\theta\,\mathrm{d}\theta$$
$$= \frac{1}{6\pi(DB_2)^{5/2}}\left[3\sqrt{2}\left(1+4DB_2+4(DB_2)^2\right)\arcsin\sqrt{\frac{2DB_2}{1+2DB_2}} - 20(DB_2)^{3/2} - 6\sqrt{DB_2}\right]. \quad (48)$$

Equation (48) represents a transcendent equation which must be solved numerically. We can considerably simplify Eq. (48) if the product of the selective-absorption coefficient, D, and the amplitude of the second-harmonics, B_2, is sufficiently large/small.

1. Case $DB_2 \gg 1$: Although with the increasing parameter, D, the second-harmonics amplitude decreases, as seen from Fig. 2, dependency of the product DB_2 on the parameter, D, represents a nonlinearly increasing function, which is apparent from Fig. 3.

We assume that the relation $DB_2 \gg 1$ and the condition (9) is satisfied.

First, we simplify the argument of the function arcsine in the relation (48),

$$\sqrt{\frac{2DB_2}{1+2DB_2}} = \frac{1}{\sqrt{1+\frac{1}{2DB_2}}} \approx \frac{1}{1+\frac{1}{4DB_2}}. \quad (49)$$

Further, for the simplified argument we can write the following expansion:

$$\arcsin\left(\frac{1}{1+\frac{1}{4DB_2}}\right) = \frac{\pi}{2} - \frac{1}{2DB_2} + \mathcal{O}\left(\frac{1}{(DB_2)^{3/2}}\right). \quad (50)$$

After substituting the expansion (50) into Eq. (48), we keep only the three dominant terms

$$B_2 = \sqrt{\frac{2}{DB_2}} - \frac{10}{3\pi DB_2} - \frac{2}{\pi DB_2} = \sqrt{\frac{2}{DB_2}} - \frac{16}{3\pi DB_2}. \quad (51)$$

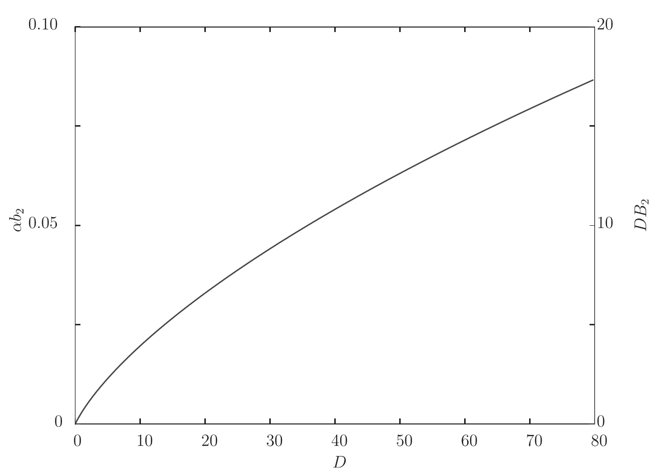

FIG. 3. Dependency of αb_2 or DB_2 on the parameter, D, for the steady state [numerically calculated using the relation (48)].

Ignoring the last term in this equality would lead to the same result as was published in Ref. 5, i.e.,

$$B_2 = \left(\frac{2}{D}\right)^{1/3}. \quad (52)$$

We modify Eq. (51) into the form

$$B_2\left(1+\frac{16}{3\pi DB_2^2}\right) = \sqrt{\frac{2}{DB_2}}. \quad (53)$$

To find an analytical solution of Eq. (53), we replace the terms in the parenthesis by the following square root:

$$1+\frac{16}{3\pi DB_2^2} \approx \sqrt{1+\frac{32}{3\pi DB_2^2}}. \quad (54)$$

Substituting the square root (54) into Eq. (53) and squaring the equation, we obtain the following cubic equation:

$$B_2^3 + \frac{32}{3\pi D}B_2 - \frac{2}{D} = 0. \quad (55)$$

This equation can be solved analytically; its only real solution is given as

$$B_2 = \frac{\left[\left(27\pi^{3/2}+\sqrt{\frac{2^{15}+3^6\pi^3 D}{D}}\right)D^2\right]^{2/3} - 32D}{3\sqrt{\pi}D\left[\left(27\pi^{3/2}+\sqrt{\frac{2^{15}+3^6\pi^3 D}{D}}\right)D^2\right]^{1/3}}. \quad (56)$$

2. *Case* $0 \le DB_2 \ll 1$: In this case, we can simplify the relation (45) into the form

$$V(\theta) = 2\cos\frac{\theta}{2}\sqrt{1+2DB_2\sin^2\frac{\theta}{2}}\,\mathrm{sign}(\theta)$$
$$\approx 2\cos\frac{\theta}{2}\left(1+DB_2\sin^2\frac{\theta}{2}\right)\mathrm{sign}(\theta). \quad (57)$$

For the wanted second-harmonics amplitude, we can write

$$B_2 = \frac{2}{\pi}\int_0^{\pi} 2\cos\frac{\theta}{2}\left(1+DB_2\sin^2\frac{\theta}{2}\right)\sin 2\theta\,\mathrm{d}\theta$$
$$= \frac{4}{\pi}\left(\frac{8}{15} - \frac{2}{3}DB_2\right). \quad (58)$$

From this equality, we obtain

$$B_2 = \frac{32}{15\pi + 40D}. \tag{59}$$

B. Solution taking into account thermo-viscous losses

The solution (45) can be generalized for thermo-viscous fluids as well in the case of $\Gamma \gtrsim 50$. Due to formal similarity of the model equation (38) with the model equation solved in Ref. 4, we can directly apply the method of derivation of an approximate solution in this paper for our purpose. The approximate solution is given by

$$V(\theta) = 2\cos\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}} \times \tanh\left[\left(\Gamma \sin\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}}\right) + \Gamma\frac{\ln\left(\sqrt{2DB_2}\sin\frac{\theta}{2} + \sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}}\right)}{\sqrt{2DB_2}}\right]. \tag{60}$$

If DB_2 is not large, both terms in the argument of the hyperbolic tangent of the solution (60) are comparable;[4] then we can simplify this solution

$$V(\theta) = 2\cos\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}} \times \tanh\left(2\Gamma \sin\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}}\right). \tag{61}$$

For large values of DB_2, the second term in the argument of the hyperbolic tangent of the solution (60) is insignificant in comparison with the first term, and then we can simplify the solution (60) into the form[4]

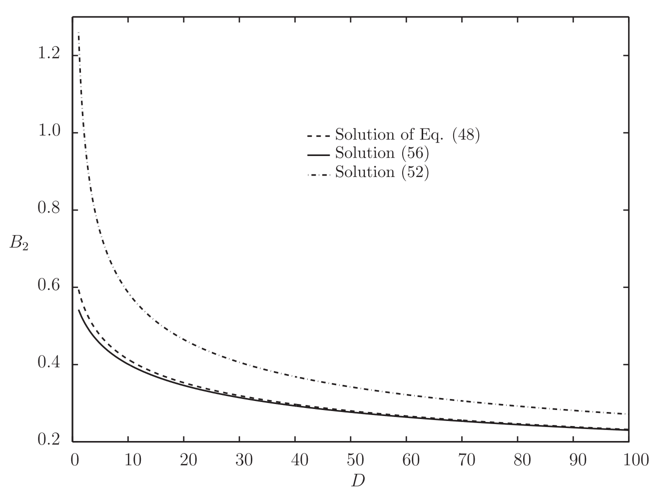

FIG. 4. Comparison of steady-state solutions for the second-harmonics B_2.

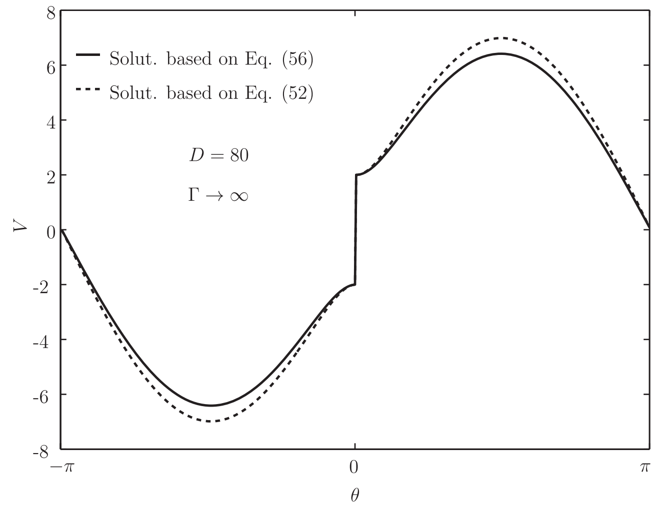

FIG. 5. Comparison of steady-state solutions based on Eqs. (52) and (56).

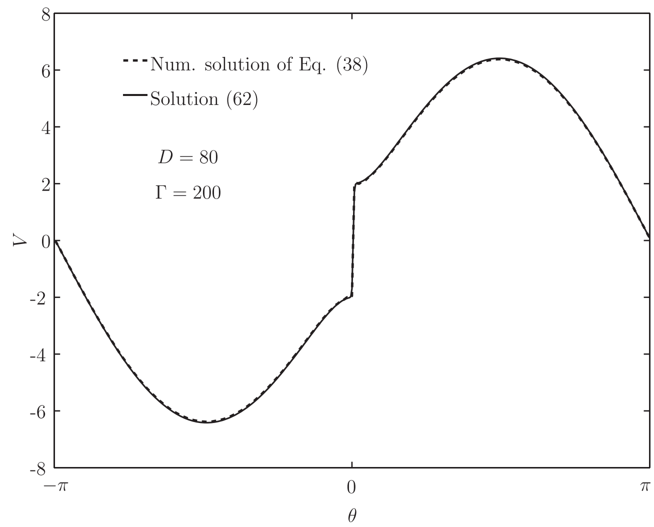

FIG. 6. Comparison of the steady-state numerical solution of Eq. (38) with the solution (62) for $D = 80$ and $\Gamma = 200$.

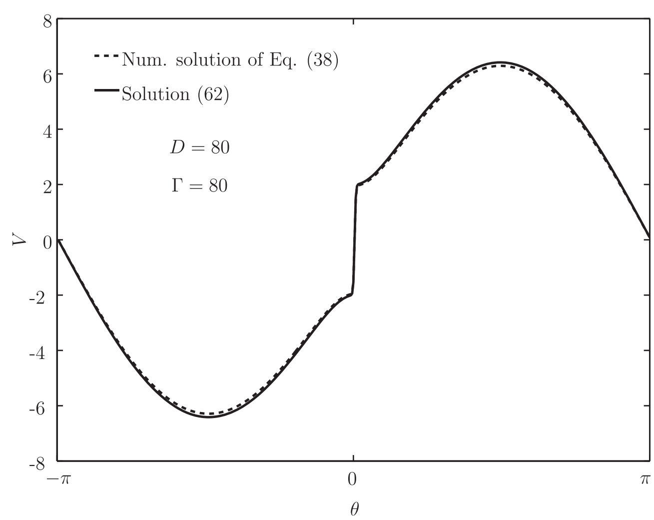

FIG. 7. Comparison of the steady-state numerical solution of Eq. (38) with the solution (62) for $D = 80$ and $\Gamma = 80$.

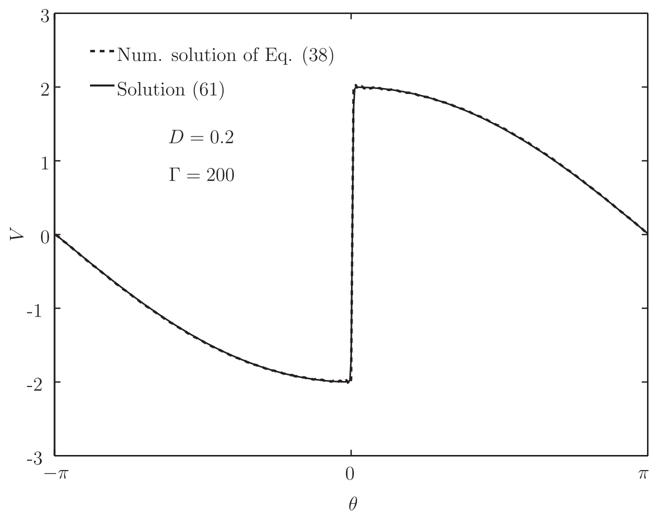

FIG. 8. Comparison of the steady-state numerical solution of Eq. (38) with the solution (61) for $D = 0.2$ and $\Gamma = 200$.

$$V(\theta) = 2\cos\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}} \times \tanh\left(\Gamma\sin\frac{\theta}{2}\sqrt{1 + 2DB_2\sin^2\frac{\theta}{2}}\right). \quad (62)$$

IV. COMPARISON OF APPROXIMATE AND NUMERICAL SOLUTIONS

The derived second-harmonics amplitude (56) and the second-harmonics amplitude (52), which was presented in Ref. 5, differ from each other considerably even for relatively high values of the parameter, D, as we can see in Fig. 4. From this figure it is possible to observe that there is a good agreement between the derived second-harmonics amplitude (56) and the numerical solution of the transcendental equation (48) even for relatively low values of the parameter, D.

The solutions of Eq. (40) are depicted in Fig. 5 for $D = 80$, $\Gamma \to \infty$. It is obvious that the solution (45), which takes the second-harmonics (52), differs significantly from the solution based on the second-harmonics amplitude (56). Figures 6 and 7 show the comparison of steady-state approximate and numerical solutions and illustrate applicability of the solutions (61) and (62) for $D = 80$. The solution (61) for the second-harmonics amplitude (58) ($D = 0.2$) together with the numerical solution are shown in Fig. 8. Again, we can observe very good agreement between the derived solution and the numerical one; both of the solutions almost coincide.

V. CONCLUSION

In this paper, we have derived the model equation that describes finite-amplitude standing waves in a cylindrical resonator with a general frequency-dependent impedance termination. The model equation is relatively complex and can be used for a number of both practical and theoretical purposes. We employed this equation for one of the possible cases of its use, which is interesting mainly from a theoretical point of view. We considered a resonator that is terminated with a second-harmonics absorber. This issue has been addressed in Ref. 5 where a steady-state approximate solution of Eq. (38) for a perfect fluid was presented. We found two solutions; whereas one of them corresponds to the solution mentioned in Ref. 5, our solution is much more accurate as it was shown in comparison with numerical solutions of the corresponding model equation.

The first solution follows from the assumption that the selective-absorption coefficient for the second-harmonics D is sufficiently large. The second approximate solution was derived for the case that the selective-absorption coefficient D is, on the contrary, small. In addition, both the solutions were improved by taking into account thermo-viscous losses. As mentioned above, the solutions were compared with the numerical ones. The comparison confirms their applicability for description of the resulting nonlinear standing waves.

ACKNOWLEDGMENTS

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