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Numerical study of the influence of the convective heat transport on acoustic streaming in a standing wave

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Within this work, acoustic streaming in an air-filled cylindrical resonator with walls supporting a temperature gradient is studied by means of numerical simulations. A set of equations based on successive approximations is derived from the Navier-Stokes equations. The equations take into account the acoustic-streaming-driven convective heat transport; as time-averaged secondary-field quantities are directly calculated, the equations are much easier to integrate than the original fluid-dynamics equations. The model equations are implemented and integrated employing commercial software COMSOL Multiphysics. Numerical calculations are conducted for the case of a resonator with a wall-temperature gradient corresponding to the action of a thermoacoustic effect. It is shown that due to the convective heat transport, the streaming profile is considerably distorted even in the case of weak wall-temperature gradients. The numerical results are consistent with available experimental data.

I. INTRODUCTION

The acoustic streaming generated by a plane standing wave in an acoustic resonator has been studied extensively in the past both by analytical and experimental methods.

Rayleigh first calculated acoustic streaming in wide two-dimensional channels whose width is much bigger than the viscous boundary layer; the corresponding formulas for wide cylindrical tubes were derived by Schuster and Matz. The thermal effects were first considered by Rott.

Within their theoretical work, Menguy and Gilbert identified a dimensionless parameter characterizing the streaming flow—the nonlinear Reynolds number

\[
\text{Re}_nl = \frac{u_0}{c_0} \left( \frac{R}{\delta_c} \right)^2 ,
\]

where \(u_0\) is the velocity amplitude of the standing wave, \(c_0\) is the speed of sound, \(R\) is the resonator radius, and \(\delta_c\) is the viscous boundary layer thickness. If \(\text{Re}_nl \ll 1\), the effect of inertia on the streaming flow can be neglected by comparison with viscous effects and we speak about the “slow” streaming; on the contrary, if \(\text{Re}_nl \gg 1\), the effect of inertia cannot be neglected anymore and we speak about the “fast” streaming. In the “slow régime,” the streaming structure does not depend on the acoustic field amplitude, in the “fast régime,” the effect of inertia, according to the theory by Menguy and Gilbert, causes a specific streaming-profile distortion which increases with the increasing value of \(\text{Re}_nl\).

Several analytical or semi-analytical models for slow streaming have been proposed for resonators of arbitrary width or radius; these models employ the methods of perturbation analysis. Numerical methods of the computational fluid dynamics have been employed for the study of the fast acoustic streaming. Within these works, it has been confirmed that if \(\text{Re}_nl \gg 1\), the streaming structure is strongly distorted and additional vortices can even appear.

Thompson et al. used laser Doppler velocimetry (LDV) for measurement of the acoustic streaming in a cylindrical tube at high values of \(\text{Re}_nl\). They found out that for higher values of \(\text{Re}_nl\), the streaming profile deviates considerably from the prediction by Rott or Menguy and Gilbert. They have shown in an experimental way that this streaming profile distortion is connected with temperature gradient developed along the resonator walls due to thermoacoustically driven heat flux. At the time, there has not been any theory at hand explaining this behaviour.

Reyt et al. compared their measurements of acoustic streaming in a cylindrical resonator using LDV with the numerical data obtained by a direct numerical integration of Navier-Stokes equations. They obtained similar experimental results (distorted streaming profiles) as Thompson et al. with a small temperature gradient along the resonator. The numerical calculations were performed at the condition of isothermal resonator walls; qualitative and overall quantitative agreement between the experimental and numerical results has been achieved so that the authors identified the inertial effects as the primary reason for the distortion of the streaming profile for high values of \(\text{Re}_nl\).

Červenka and Bednár have recently shown that (a) the slow-streaming profile is particularly sensitive to the temperature variation in the direction perpendicular to the resonator axis, (b) this variation can be caused by the wall-temperature gradient if the resonator is wide enough, and (c) additional outer streaming cells can appear like in the case of the fast streaming in resonators with isothermal walls. The underlying mathematical model does not take into account acoustic streaming as a means of heat transport which means that the results are only valid for small streaming velocities.

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In a very recent work, Daru et al. argue, based on the results of numerical experiments, that the inertial effects cannot be leading mechanism of streaming structure distortion observed in experiments and direct numerical simulations; they consider the role of the nonlinear interactions between the streaming flow and the acoustic field.

Within this work, we show by means of numerical simulations that even for moderate values of $\text{Re}_{nl}$, the streaming profile can be considerably distorted from the sinusoidal one predicted by Rayleigh’s theory, if there is even weak temperature gradient along the resonator walls. This acoustic-field-amplitude-dependent distortion is not caused by the effect of the fluid inertia, but it is connected with the streaming-driven convective heat transport in a fluid with a mean temperature gradient. The streaming profile distortion described within this work is of the same type as it has been found in experiments.\textsuperscript{11,12,15,16}

The results described within this work cannot be obtained using the previous theoretical models dealing with acoustic streaming in temperature-inhomogeneous fluids,\textsuperscript{3,5,6,13} as these models do not capture the effect of the acoustic-streaming-driven convective heat transport.

Section II of this paper describes the theoretical model and the numerical procedure; examples of numerical results are presented in Sec. III. A discussion and explanation of the observed effects are given in Sec. IV; Sec. V then concludes the paper.

II. THEORETICAL MODEL

A. Model equations

The study of the acoustic streaming starts with the Navier-Stokes equations, which can be written as\textsuperscript{17}

\[
\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u}, \quad (2a)
\]

\[
\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \mathbf{\sigma} + \mathbf{f}, \quad (2b)
\]

\[
\rho c_p \frac{dT}{dt} - \alpha T \frac{d\rho}{dt} = \nabla \cdot \left( \kappa \nabla T \right) + \tau \cdot \nabla \mathbf{u}, \quad (2c)
\]

where $\rho$ is the fluid density, $\mathbf{u}$ is the velocity vector, $T$ is the temperature, $p$ is the pressure, and $\mathbf{f}$ is the body force density, as the resonator is assumed to be driven by an inertial force (by entire-body shaking); it reads $\mathbf{f} = -\rho \mathbf{a}(t)$, where $\mathbf{a}(t)$ is the resonator acceleration. Within this work, the effect of the gravity on fluid flow is not taken into account. Further, $c_p$ is the specific heat capacity at constant pressure, $\alpha$ is the isobaric coefficient of volumetric thermal expansion, and $\kappa$ is the coefficient of thermal conduction. The total stress tensor $\mathbf{\sigma}$ is defined as

\[
\mathbf{\sigma} = -p \mathbf{I} + \tau = -p \mathbf{I} + \mu \left[ \nabla \mathbf{u} + \left( \nabla \mathbf{u} \right)^T \right] - \mu_t \left( \frac{2}{3} \mathbf{I} \right) \left( \nabla \cdot \mathbf{u} \right) \mathbf{I}, \quad (3)
\]

where $\tau$ is the viscous stress tensor, $\mu$ is the shear viscosity, $\nabla = \mu_B / \mu$, where $\mu_B$ is the bulk viscosity and $\mathbf{I}$ is the identity matrix. The material parameters are assumed to be temperature-dependent, i.e., $\mu = \mu(T)$, $\kappa = \kappa(T)$, $c_p = c_p(T)$.

In Eq. (2), the material derivative is defined as $d(-)/dt = \partial(-)/\partial t + (\mathbf{u} \cdot \nabla)(-)$.

The equations for the first-order acoustic field quantities can be derived from the linearised state equation $p_a / \rho_m = p_T / T_m + \rho_a / \rho_m$.

The equations for the time-averaged quantities can be derived from the linearised state equation $p_a / \rho_m = p_T / T_m + \rho_a / \rho_m$. The equations (3a), (3b) can be used for calculation of the primary acoustic field.

The equations for the one-period average of relation (4), we get

\[
\langle \phi \rangle = \phi_m(r, \tau) = \frac{1}{T_s} \int_{t_n}^{t_n + T_s} \phi(r, t) dt = \phi_0 + \langle \phi_n \rangle, \quad (5)
\]

where $T_s = 2\pi/\omega$ and $t_s$ represents the slow time related to the large-time-scale phenomena.\textsuperscript{18}

The equations for the first-order acoustic field quantities can be derived from the linearised state equation $p_a / \rho_m = p_T / T_m + \rho_a / \rho_m$. Equation (6) can be used for calculation of the primary acoustic field.

The equations for the one-period average of relation (4), we get

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}_m) = 0, \quad (6a)
\]

\[
\rho_m \frac{d\mathbf{u}_m}{dt} = -V \cdot \left\{ -p_m I + \mu_m \left[ \nabla \mathbf{u}_m + \left( \nabla \mathbf{u}_m \right)^T \right] \right\} - \mu_m \left( \frac{2}{3} \right) \nabla \cdot \mathbf{u}_m \mathbf{I} = -\rho_m \mathbf{a}, \quad (6b)
\]

\[
\rho_m c_{pm} \frac{dT_m}{dT_s} + \mathbf{u}_m \cdot \nabla T_m - \frac{\partial p_m}{\partial T} - \nabla \cdot \left( \kappa_m \nabla T_m \right) = 0, \quad (6c)
\]

where $c_{pm} = c_p(T_m)$, $\mu_m = \mu(T_m)$, together with the linearised state equation $p_a / \rho_m = p_T / T_m + \rho_a / \rho_m$. Equation (6) can be used for calculation of the primary acoustic field.
where the source terms $M$, $F$, and $Q$ read

$$M = -\nabla \cdot (\rho_s u_s), \quad (8a)$$

$$F = -\left( \rho_s (a + \frac{\partial u_s}{\partial t}) - \rho_m ((u_s \cdot \nabla)u_s) + \nabla \left( \frac{\mu_m b_m}{T_m} T_a \right) \times \left[ \nabla u_s + (\nabla u_s)^T - \left( \frac{2}{3} + \bar{V} \right) (\nabla \cdot u_s) I \right] \right), \quad (8b)$$

$$Q = -\rho_m c_p \left( \frac{\partial T_a}{\partial t} \right) - \rho_m c_p (u_s \cdot \nabla T_a) - c_p (\rho_s u_s) \cdot \nabla T_m
+ (u_s \cdot \nabla p_s) + \nabla \left( \frac{\kappa_m b_m}{T_m} V T_a \right)
+ \mu_m \left[ \nabla u_s + (\nabla u_s)^T - \left( \frac{2}{3} - \bar{V} \right) (\nabla \cdot u_s) I \right] : \nabla u_s \right). \quad (8c)$$

Here, $\{f\} = \Re \{\tilde{f} \tilde{\phi}^*\}/2$; the tildes represent the complex amplitudes of the corresponding quantities and the asterisk stands for the complex conjugate. Further, the following relation was used:

$$\mu(T) \approx \mu(T_m + T_a) \approx \mu_m + \frac{\mu_m b_m}{T_m} T_a,$$

where $b_m = T_m (\partial \mu/\partial T)_{T_m}/\mu_m$, and

$$\kappa(T) \approx \kappa(T_m + T_a) \approx \kappa_m + \frac{\kappa_m b_m}{T_m} T_a,$$

where $b_m = T_m (\partial \kappa/\partial T)_{T_m}/\kappa_m$. Acoustic-temperature-induced variation of the heat capacity was not taken into account in Eq. (7c) because it is weaker than the variation of the viscosity or thermal conduction coefficients.

The source terms [Eq. (8)] in Eq. (7) as a consequence of employing the method of successive approximations; they comprise nonlinear combinations of the first-order acoustic field quantities. Namely, $M$ is the mass source, $F$ represents the excitation force (Reynolds stress, the force caused by the dependence of viscosity on acoustic temperature), and $Q$ represents a heat source.

In the steady state, the averaged quantities do not depend on the slow time and thus Eqs. (7a), (7b), and (7c) reduce into

$$\nabla \cdot (\rho_m u_m) = M, \quad (9a)$$

$$\rho_m u_m \cdot \nabla u_m - \nabla \cdot \left\{ -p_m I + \mu_m \left[ \nabla u_m + (\nabla u_m)^T \right] - \frac{2}{3} \bar{V} \left( \nabla \cdot u_m \right) I \right\} = F, \quad (9b)$$

$$\rho_m c_p u_m \cdot \nabla T_m - u_m \cdot \nabla p_m - \nabla \cdot (\kappa_m \nabla T_m) = Q. \quad (9c)$$

Within this model, we do not take into account the fast time-varying products of the nonlinear interactions (higher harmonics), which can cause the distortion of the time-harmonic primary acoustic field. This seems not to have any significant impact on the results, as a considerable streaming-profile distortion has been observed within the experiments for high values of $Re_m$ and time-harmonic acoustic field.

The structure of the acoustic streaming can be easily visualised in the following way. If we introduce the averaged mass transport density $M_m = \rho_m u_m + \langle \rho_s u_s \rangle = \rho_m U_m$, where $U_m$ is the averaged mass transport velocity, the continuity Eq. (9a) can be rewritten as

$$\nabla \cdot M_m = 0. \quad (10)$$

As the steady averaged mass transport density field is divergence-free, the stream function $\psi$ can be introduced such that in axi-symmetric cylindrical coordinates, it holds

$$M_{mz} = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad M_{mr} = -\frac{1}{r} \frac{\partial \psi}{\partial z}.$$ From here, we can calculate the stream function as

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial}{\partial r} (r M_{mz}) - \frac{\partial}{\partial z} (r M_{mr}). \quad (11)$$

The contours of the stream function represent the streamlines.

**B. Numerical solution of the model equations**

Equations (6) and (7) represent one set of equations which must be solved simultaneously to capture properly the effect of acoustic-streaming-driven convective heat transport on the streaming structures. The source terms [Eq. (8)] for Eq. (7) are calculated using the first-order quantities obtained by solving Eq. (6), the mean values $T_m$, $\rho_m$, and $p_m$ in Eq. (6) are calculated using Eq. (7).

The model equations were solved numerically using finite-element-method software COMSOL Multiphysics; axi-symmetric geometry was used, see Fig. 1. Equation (6) is implemented in the Acoustic Module’s Linearised Navier-Stokes Interface; the equations were solved in the frequency domain with no-slip and isothermal ($T_i = 0$) boundary conditions at the resonator walls. The term $-\rho_m a$ was used as a volume source in the momentum Eq. (6b); the driving acceleration is assumed to have non-zero only the $z$-component and to vary sinusoidally in time with an angular frequency $\omega$. Equation (7) is implemented in CFD Module’s Non-Isothermal Laminar Flow Interface; the equations were solved using the stationary as well as the time-dependent solver. The boundary conditions at the resonator walls were implemented as no-slip and isothermal ones with prescribed wall temperature ($T_m = T_w$).

The resonance frequency was determined by a frequency sweep for a small driving acceleration for which the convective heat transport by the streaming could be neglected.

**FIG. 1.** (Color online) Geometry of the problem.
The Poisson’s Eq. (11) was solved employing Mathematics/Classical PDEs Interface with the homogeneous Dirichlet boundary conditions.

The computational mesh was constructed in order to properly resolve the boundary layer along the walls; mapped (structured) mesh with refinement along the walls was found to work well; the independence of numerical results on the mesh density was checked.

For the numerical results presented below, the total number of used mesh-points was $N_r \times N_z = 100 \times 200$, where 15 regularly-spaced mesh-points discretized three-times the viscous boundary layer thickness, and mesh-points with linearly increasing separation-distances were used in the inner volume.

III. NUMERICAL RESULTS

A. Parameters of the numerical simulations

Within the numerical simulations, the acoustic streaming was calculated in a resonator represented by a cylindrical tube with rigid end-caps, length $L = 30$ cm, radius $R = 1.5$ cm, filled with air at normal atmospheric pressure. The temperature-dependent values of the material parameters $\mu$, $\kappa$, and $c_p$ are implemented in the material database of COMSOL Multiphysics, the ones for air were used here. For ambient temperature $T_0 = 20$ $^\circ$C, the resonance frequency was calculated to be $f_{res} = 569.2$ Hz and the ratio of the resonator radius to the viscous boundary layer thickness $R/\delta_v = 163.4$.

The resonant cavity is assumed to have imposed temperature of walls

$$T_w = \frac{\Delta T_w}{2} \left[ 1 + \cos \left( \frac{2\pi z}{L} \right) \right] + T_{w0}, \quad (12)$$

where $T_{w0}$ is the minimum temperature of the resonator walls and $\Delta T_w$ is the maximum wall-temperature difference. Formula (12) serves as the boundary condition for Eq. (7c) for calculation of the fluid mean temperature $T_m$. In all the following cases, $T_{w0} = 20$ $^\circ$C and temperatures $\Delta T_w$ differ.

Formula (12) models\textsuperscript{12} the steady-state wall temperature distribution due to the thermoacoustic heat transport. In the standing wave, there exists a time-averaged heat flux between the walls and the inner fluid which transports the heat from the acoustic-velocity antinodes toward the acoustic-velocity nodes.\textsuperscript{19,20} This flux, which is proportional to the square of the acoustic velocity amplitude,\textsuperscript{19} is balanced by the heat conduction within the walls and the heat flux to the outer environment; the resulting wall-temperature gradient thus depends on particular external factors, and that is why it is prescribed by Eq. (12) here.

All the numerical results are presented in the conditions of the first resonance which was searched for by a frequency sweep by maximization of the acoustic velocity amplitude in the resonator.

B. The case of $\Delta T_w = 8$ $^\circ$C

Within this subsection, we analyse the case of a relatively large wall-temperature difference of $\Delta T_w = 8$ $^\circ$C (this wall-temperature difference was reported to be thermoacoustically induced in experiment\textsuperscript{15}).

Figure 2 shows the distribution of steady-state averaged mass transport velocity along the resonator axis for individual values of Re\textsubscript{nl} both normalized to the Rayleigh velocity $U_R = (3/8)u_0/c_0$ (top panel) and non-normalized (bottom panel), where $u_0$ is the maximum axial acoustic velocity amplitude on the resonator axis, and $c_0$ is the speed of sound. Within this work, we do not exclusively understand Re\textsubscript{nl} as a measure of flow-inertia effect on the streaming field; we employ it as an established parameter related to the streaming velocity (according to the Rayleigh’s theory, streaming velocity should be proportional to Re\textsubscript{nl}).

In the case of Re\textsubscript{nl} = 0.001, the streaming velocity has such a small value that the convective heat transport is negligible. The normalized streaming velocity along the axis reaches the maximum value of $U_{\text{m}n}/U_R = 1.009$; the distribution slightly differs from the sinusoidal one,\textsuperscript{2} which is also in line with the observations made earlier.\textsuperscript{13} If the streaming velocity increases, its distribution quickly gets distorted, and the value of the normalized streaming velocity in maxima decreases; the maxima are shifted toward the acoustic velocity nodes (resonator end-walls). For example, for Re\textsubscript{nl} = 0.2, the maximum value of $U_{\text{m}n}/U_R = 0.771$. With further increasing Re\textsubscript{nl}, the normalised streaming velocity between the velocity nodes and antinodes further decreases; it reaches zero value for Re\textsubscript{nl} $\approx$ 2.4 and its direction reverses for higher values of Re\textsubscript{nl}. Considering the non-normalised streaming velocity, the situation looks somewhat more complex; while its value increases with Re\textsubscript{nl} monotonically near the acoustic velocity nodes and antinode, between them, it first increases, then reaches maximum, decreases and then changes its sign. It can be seen in both versions of the figure that the incremental change of the streaming velocity is bigger for lower values of Re\textsubscript{nl} than for higher ones.

Figure 3 shows the distribution of the $z$-component of the steady-state averaged mass transport velocity along the $r$-axis for $z = L/4 = 7.5$ cm, both normalized to the Rayleigh
Again, it can be observed that for higher values of $\text{Re}_{nl}$ it deviates considerably from the “parabolic” distribution.\textsuperscript{2,3} It is interesting to notice that despite the streaming structure is sensitive to the value of $\text{Re}_{nl}$ in the inner parts of the resonator, it is more or less unaffected near the walls (the normalized velocity does not depend on $\text{Re}_{nl}$; non-normalized velocity increases proportionally to $\text{Re}_{nl}$).

Figure 4 shows the streamlines for the case of $\text{Re}_{nl} = 0.001$ and $\text{Re}_{nl} = 3.0$. If the streaming velocity is small (top panel), two outer vortices can be observed as expected.\textsuperscript{8} The inner vortices are not appreciable because of the high value of the ratio $R/\delta_w$. For higher streaming velocity (bottom panel) the streamlines are distorted, and two weak additional outer vortices appear near the resonator axis, which is the cause of the reversion of the streaming velocity at the resonator axis seen in Figs. 2 and 3.

In order to get some conception of the time scale of the processes connected with the convective heat transport by the acoustic-streaming, Fig. 5 shows the time evolution of the averaged mass transport velocity at $z = L/4$ on the resonator axis for individual values of $\text{Re}_{nl}$. As an initial condition, the steady-state solution of Eqs. (6)-(9) was used, where the convective term $u_m \cdot \nabla T_m$ was omitted in the energy Eq. (9c). It can be seen that for small values of $\text{Re}_{nl}$, the streaming velocity monotonically decreases toward the steady-state value; for the higher values, there is an overshoot with possible temporary direction reversion. After the initial faster development, the streaming velocity slowly reaches the steady-state in less than ca. 20 s.

Figure 6 shows the distribution of the steady-state mean temperature $T_m$ along the resonator axis for various values of $\text{Re}_{nl}$. It can be observed that with increasing $\text{Re}_{nl}$, mean temperature in the central part of the resonator (especially in the regions where the temperature gradient reaches the highest values) increases as the streaming convects heat along the axis from the acoustic velocity nodes toward the acoustic velocity antinode. An effect of saturation can be observed in the figure—despite that there is appreciable temperature difference between the cases of $\text{Re}_{nl} = 0.001$ and $\text{Re}_{nl} = 1$, it is only negligible for the cases of $\text{Re}_{nl} = 2$ and $\text{Re}_{nl} = 3$. This behaviour can be attributed to the decrease and reversion of the streaming velocity seen in Figs. 2 and 3.

**C. Different values of $\Delta T_w$**

Similar behaviour as the one described in Sec. III B can be observed in the case of different wall temperature differences $\Delta T_w$. 

In order to get some conception of the time scale of the processes connected with the convective heat transport by the acoustic-streaming, Fig. 5 shows the time evolution of the averaged mass transport velocity at $z = L/4$ on the resonator axis for individual values of $\text{Re}_{nl}$. As an initial condition, the steady-state solution of Eqs. (6)-(9) was used, where the convective term $u_m \cdot \nabla T_m$ was omitted in the energy Eq. (9c). It can be seen that for small values of $\text{Re}_{nl}$, the streaming velocity monotonically decreases toward the steady-state value; for the higher values, there is an overshoot with possible temporary direction reversion. After the initial faster development, the streaming velocity slowly reaches the steady-state in less than ca. 20 s.

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**FIG. 3.** (Color online) Normalized (top panel) and non-normalized (bottom panel) averaged mass transport velocity along the line $z = L/4$, $\Delta T_w = 8$ °C, $\text{Re}_{nl} = 0.001, 0.2, 0.4, \ldots, 2.8, 3.0$; steady state.

**FIG. 4.** (Color online) Averaged mass transport velocity and streamlines for $\Delta T_w = 8$ °C and different values of $\text{Re}_{nl}$; steady state.

**FIG. 5.** (Color online) Time evolution of the averaged mass transport velocity at $z = L/4$ on the resonator axis; $\Delta T_w = 8$ °C, and $\text{Re}_{nl} = 0.001, 0.2, 0.4, \ldots, 2.8, 3.0$.

**FIG. 6.** (Color online) Mean temperature along the resonator axis $\Delta T_w = 8$ °C, and $\text{Re}_{nl} = 0.001, 1.0, 2.0, 3.0$; steady state.
For example, Fig. 7 shows the steady-state distribution of the averaged mass transport velocity along the resonator axis for $\Delta T_w = 4{\degree}C$ and different values of $Re_{nl}$. The streaming velocity shows the same kind of distortion as in the previous case (see Fig. 2); higher values of $Re_{nl}$ would cause the streaming velocity reversion as we were not able to reach the algorithm convergence for these conditions.

Figure 8 shows the distribution of the steady-state normalized averaged mass transport velocity along the resonator axis for $Re_{nl} = 1$ and different values of the wall temperature difference $\Delta T_w = 0{\degree}C, 0.5{\degree}C, 1{\degree}C, 2{\degree}C, 4{\degree}C, 8{\degree}C$, and $16{\degree}C$. The figure reveals that the deviation from the sinusoidal distribution of the streaming velocity behaves similarly as in the case of constant wall-temperature difference and increasing value of $Re_{nl}$—see Figs. 2 and 7. It can be observed that for the high wall-temperature differences, the streaming velocity pattern is considerably distorted even for small values of $Re_{nl}$.

The sensitivity of the acoustic streaming velocity on the wall-temperature difference is depicted in Fig. 9 where $U_{m}/U_R$ at $z = L/4 = 7.5$ cm on the resonator axis is plotted as a function of $Re_{nl}$ for individual values of $\Delta T_w$. For $\Delta T_w = 0{\degree}C$, the streaming velocity decreases with increasing $Re_{nl}$ only weakly due to the effect of streaming fluid inertia, as it has been shown by Menguy and Gilbert. The higher the maximum wall-temperature difference $\Delta T_w$, the stronger the dependence on $Re_{nl}$. It is interesting that the dependence is quite appreciable even for small values of $\Delta T_w$, for example, for $Re_{nl} = 4$ and $\Delta T_w = 0.5{\degree}C$, the value of $U_{m}/U_R$ reaches only $70\%$ of the value for $\Delta T_w = 0{\degree}C$.

IV. DISCUSSION

It has been shown recently that the acoustic streaming structure is particularly influenced by the mean temperature gradient in the direction perpendicular to the resonator axis. If $\Delta T_m(z) = T_m(0, z) - T_m(R, z) < 0$, acoustic streaming near the resonator axis is locally supported, if $\Delta T_m(z) > 0$, it is locally opposed which may even result in the development of additional outer vortices. If the streaming velocity is small enough for the convective heat transport to be effective, it can be approximately written [for the wall-temperature distribution Eq. (12)]

$$\Delta T_m(z) \approx -\frac{\pi^2 R^2 \Delta T_w}{2L^2} \cos \left(\frac{2\pi z}{L}\right).$$

This explains a small departure from the sinusoidal streaming velocity distribution seen in Figs. 2 and 7 (the cases of $Re_{nl} = 0.001$).

If the resonator driving is increased, $Re_{nl}$ increases; acoustic streaming becomes a more effective means of the heat transport which results in the re-distribution of the mean fluid temperature. Acoustic streaming convects the heat along the resonator axis from the warmer areas near the end-walls toward the resonator centre, see Fig. 10, which explains the temperature increase as seen in Fig. 6.

However, this heat transport also increases the value of $\Delta T_m(z)$ in the resonator central part, see an example in Fig. 11, leading to the opposition to the acoustic streaming and reducing its effectiveness in convecting the heat. This effect manifests itself by decreasing incremental temperature change with increasing $Re_{nl}$. This feedback behaviour explains the distortion of the streaming structure in temperature-inhomogeneous fluids and its dependence on $Re_{nl}$ through the convective term $u_m \cdot \nabla T_m$ in the energy equation.
inertial effects on the streaming fluid flow are small\(^4\) and the temperature inhomogeneity. The situation captured in Fig. 8 is the behaviour observed in experiments; \(^{11,12,15}\) in strong distortions depends on the temperature variation along the resonator walls, which causes the suppression of acoustic streaming, its ability to convect the heat, and to further increase the temperature difference.

**V. CONCLUSIONS**

A set of equations for calculation of the acoustic streaming in resonators with walls supporting a temperature gradient has been derived from the Navier-Stokes equations. The model equations are based on successive approximations; they take into account the acoustic-streaming-driven convective heat transport; as time-averaged secondary-field quantities are directly calculated, there is no need for an excessive amount of computational effort for their integration. The proposed computational procedure has been implemented in COMSOL Multiphysics.

A parametric study has been conducted for the case of an air-filled cylindrical resonator with walls with a temperature distribution corresponding to the one caused by the thermoacoustic effect.

It has been shown that even in the case of relatively weak wall-temperature gradients, the acoustic-streaming-driven convective heat transport is responsible for a considerable distortion of streaming profile; this effect is much stronger than the effect of the fluid inertia on the streaming flow. The streaming profiles observed within this work are consistent with the available experimental data.

Based on the results obtained within this work, we arrive at the conclusion that for a good quantitative agreement between experimental and theoretical data, the thermal effects must be comprised within the model describing acoustic streaming, and even slight wall-temperature gradients cannot be ignored anymore.

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\(^{1}\)Lord Rayleigh, “On the circulation of air observed in Kundt’s tubes, and on some allied acoustical problems,” Philos. Trans. R. Soc. London 175, 1–21 (1884).


