Propagation of nonlinear acoustic plane waves in an elastic gas-filled tube

Michal Bednarik, Milan Cervenka, and OAS

Citation: The Journal of the Acoustical Society of America **126**, 1681 (2009); doi: 10.1121/1.3203936
View online: http://dx.doi.org/10.1121/1.3203936
View Table of Contents: http://asa.scitation.org/toc/jas/126/4
Published by the Acoustical Society of America

Propagation of nonlinear acoustic plane waves in an elastic gas-filled tube

Michal Bednarik and Milan Cervenka[a)]
Faculty of Electrical Engineering, Czech Technical University in Prague, Technicka 2, 166 27 Prague, Czech Republic

(Received 26 January 2009; revised 9 June 2009; accepted 17 July 2009)

This paper deals with modeling of nonlinear plane acoustic waves propagating through an elastic tube filled with thermoviscous gas. A description of the interactions between gas and an elastic tube wall is carried out by the continuity equation of a wall velocity. Simplification on the basis of the local reaction assumption enables to model an acoustic treatment on the tube wall by using a wall impedance. Because there are considerable losses due to wall friction, the influences of the acoustic boundary layer were also considered. Using certain assumptions a special form of the Burgers equation was derived which enables to describe the propagation of nonlinear waves in the elastic tube. This model equation takes into account nonlinear, dissipative, and dispersion effects which compete each other. Characteristic lengths of the supposed effects and numerical results with respect to the source frequency were used for a qualitative analysis of the model equation. Applicability of this model equation was demonstrated by series of measurements. By application of the long-wave approximation the Korteweg–de Vries–Burgers and Kuramoto–Sivashinsky equations were derived from the modified Burgers equation.
 [DOI: 10.1121/1.3203936]

PACS number(s): 43.25.Cb, 43.20.Mv [OAS] Pages: 1681–1689

I. INTRODUCTION

The problems concerning with the interactions between acoustic oscillations in a fluid-filled tube and the vibrations of its wall have received the attention of many investigators. Since time of Young who first found the pulse wave speed in human arteries, a number of scientists dealt with propagation of acoustic waves through elastic tubes filled with fluids. Their works differ each other by various assumptions, in particular, as far as media, wave modes, wave amplitudes, and tube walls are concerned.

Fay *et al.*[1] carried out an analytical and experimental investigation of a water-filled acoustic impedance tube. Their work was motivated by the fact that water-filled tube walls cannot be assumed to be rigid for acoustic waves. The paper by Jacobi[2] was also focused on the problems regarding the sound transmission through tubes filled with ideal liquid; however, in addition he considered the higher wave modes. In contrast to the above mentioned authors who considered only inviscid fluids, Morgan and Kiely[3] took into account also viscosity of the liquid and internal damping in the tube wall. Sondhi[4] for investigation of wave propagation in a lossy vocal tract used an approach which was based on a local wall admittance model. Guelke and Bunn[5] presented work which deals with application of the transmission line theory to linear acoustic wave propagation through tube with yielding walls. They limited themselves to vibrations only in the radial direction and considered only the breathing circumferential motion. In a similar spirit, Fredberg *et al.*[6] modeled mechanic oscillations of the respiratory system at high frequencies. Elvira-Segura[7] dealt with the study of speed and attenuation of an acoustic wave propagating inside a cylindrical elastic tube filled with a viscous liquid. This author extended influence of the liquid viscosity by the boundary layer effects. Because the flaws in the roundness of a tube induce coupling between the structural and acoustic modes which do not exist in the case where the cross-section is perfectly circular, Pico and Gautier[8] presented a model which allows us to take into account the small imperfections in the tube circularity. Gautier *et al.*[9] besides a well-arranged bibliographical review presented a study on cylindrical membranes submitted to a static tension. The authors of the above-cited papers supposed only the thin-walled tubes. Grosso[10] considered the exact longitudinal and shear wave equations for the multimode axial acoustic propagation in tubes of arbitrary wall thickness filled with inviscid liquid. Nonlinear effects were taken into account, e.g., by Yomosa[11] who described the propagation of weakly nonlinear waves in an infinitely long distensible thin-walled elastic tube filled with an ideal fluid. Erbay and Dost[12] investigated the propagation of weakly nonlinear waves in an infinitely long nonlinear viscoelastic thin tube filled with incompressible, inviscid fluid. By means of the long-wave approximation they derived a number of nonlinear evolution equations representing various regimes. Kamakura and Kumamoto[13] presented a work which concerned with investigation of nonlinear plane acoustic waves through an elastic tube. The authors assumed that an elastic tube wall reacts locally to the inner pressure. On the basis of their assumptions they derived a model equation. Validation of the model equation was justified experimentally. Unfortunately, the comparison of theoretical and experimental results was not presented in a way enabling to evaluate measure of the validation of their model equation

[a)]Author to whom correspondence should be addressed. Electronic mail: bednarik@fel.cvut.cz

rigorously. Though the problems concerning propagation of acoustic waves through tubes were widely studied by many authors, investigation seems to be missing which would simultaneously include most of effects that influence acoustic waves. A model equation, which takes into account more effects at the same time, can offer qualitatively new results. This fact determines the main object of this work.

Hence this work is focused on the description of nonlinear plane acoustic waves propagating through an elastic tube filled with thermoviscous gas. For this purpose, a special form of the Burgers equation was derived on the basis of the local wall reaction hypothesis. This model equation extends the standard Burgers equation by terms which represent dispersion and dissipative effects caused by the wall elasticity and acoustic (Stokes) boundary layer which plays an important role in the course of wave forming, in particular, for lower frequencies. Theoretical results were verified experimentally. The Korteweg–de Vries–Burgers (KdVB) and Kuramoto–Sivashinsky equations are derived from the modified Burgers equation by means of the long-wave approximation. These equations are supplemented by the term which takes into account boundary layer effects.

Section II is dedicated to derivation of the model equations. Analysis of numerical solutions of the modified Burgers equation is presented in Sec. III. Then in Sec. IV we compare theoretical and experimental data to justify applicability of the model equations.

II. MODEL EQUATIONS AND DISPERSION RELATIONS

A. Derivation of the modified Burgers equation for a gas-filled elastic tube

If we take into account the acoustic boundary layer effects we can write the following one-dimensional continuity equation (see Ref. 14):

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\frac{2}{r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\rho\int_0^\infty \frac{\partial v(x,t-\chi)}{\partial x}\frac{d\chi}{\sqrt{\chi}}. \tag{1}$$

Here x is space coordinate in the direction of the tube axis, t is time, $\rho=\rho'+\rho_0$ is density of fluid, where ρ' is the acoustic density and ρ_0 is density corresponding to the equilibrium fluid state, v is the acoustic velocity, ν is the kinematic viscosity, $\Pr=\rho_0\nu c_p/\varkappa$ is the Prandtl number, $\gamma=c_p/c_V$ is the ratio of specific heats (c_p and c_V are the specific heats under constant pressure and volume, respectively), $\varkappa$ is the coefficient of heat conductivity, and r_0 is an equilibrium tube inner radius.

Using the continuity equation (1) is conditioned by the following relations:[15,16]

$$\delta \ll \lambda, \quad \delta \ll r_0,$$

where δ is a boundary layer thickness and λ is a wavelength.

Further, when it is satisfied

$$\frac{2}{r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\sim\mu,$$

where $\mu \ll 1$ is a small dimensionless parameter (the peak Mach number of the source), then within the scope of the second order nonlinear theory we can replace the density $\rho=\rho'+\rho_0$ by an ambient fluid density ρ_0 in Eq. (1). After the replacement and using the relation between the acoustic velocity and the velocity potential $v=\partial\varphi/\partial x$ Eq. (1) can be written in the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\frac{2\rho_0}{r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\int_0^\infty \frac{\partial^2\varphi(x,t-\chi)}{\partial x^2}\frac{d\chi}{\sqrt{\chi}} =\frac{2\rho_0}{r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\frac{\partial^2}{\partial x^2}\int_0^\infty \varphi(x,t-\chi)\frac{d\chi}{\sqrt{\chi}}. \tag{2}$$

With help of the linear one-dimensional wave equation

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2},$$

it is possible to rewrite Eq. (2)

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\frac{2\rho_0}{c_0^2 r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\frac{\partial^2}{\partial t^2}\int_0^\infty \varphi(x,t-\chi)\frac{d\chi}{\sqrt{\chi}} =\frac{2\rho_0}{c_0^2 r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\int_0^\infty \frac{\partial^2\varphi(x,t-\chi)}{\partial t^2}\frac{d\chi}{\sqrt{\chi}}. \tag{3}$$

The equality $\partial^2\varphi/\partial t^2=\partial^2\varphi/\partial\chi^2$ enables us to express Eq. (3) as

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\frac{2\rho_0}{c_0^2 r_0}\sqrt{\frac{\nu}{\pi}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\int_0^\infty \frac{\partial^2\varphi(x,t-\chi)}{\partial \chi^2}\frac{d\chi}{\sqrt{\chi}}. \tag{4}$$

The integral on the right hand side (rhs) of Eq. (4) can be rewritten as

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\partial^2\varphi(x,t-\chi)}{\partial \chi^2}\frac{d\chi}{\sqrt{\chi}}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^t \frac{\partial^2\varphi(x,\chi)}{\partial \chi^2}\frac{d\chi}{\sqrt{t-\chi}}. \tag{5}$$

The rhs of Eq. (5) represents the fractional derivative (A3) (see, e.g., Refs. 17, 15, and 16). We can express Eq. (3) as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\frac{2\rho_0 B}{c_0^2}\frac{\partial^{3/2}\varphi}{\partial t^{3/2}}, \tag{6}$$

where

$$B=\frac{\sqrt{\nu}}{r_0}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right) \tag{7}$$

is the boundary layer parameter.

If we suppose that acoustic waves propagate through a tube with an elastic wall (a variable cross-section) and we do not take into account the boundary layer effects then we can express the continuity equation as

$$\frac{\partial(\rho S)}{\partial t}+\frac{\partial(\rho S v)}{\partial x}=0, \quad (8)$$

where $S=S(x,t)$ is an inner tube cross-section. Further, Eq. (8) can be modified into the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}+\frac{\rho}{S}\frac{dS}{dt}=0, \quad (9)$$

where the operator d/dt is given as

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}. \quad (10)$$

If the elastic tube wall is assumed to be locally reacting (see, e.g., Refs. 18 and 5) then we can consider the inner cross-section S only as a function of time, i.e., $S=S(t)=\pi r^2(t)$, where $r(t)=r_0+r'(t)$ is a total inner tube radius and r' represents a change in r.

If we use the fact that $r'\ll r_0$, then it is possible to use the following simplification:

$$\frac{1}{S}\frac{dS}{dt}\simeq\frac{1}{S_0}\frac{dS}{dt}=\frac{1}{\pi r_0^2}\frac{d(\pi r^2)}{dt}=\frac{2}{r_0}\frac{dr}{dt}=\frac{2}{r_0}v_w, \quad (11)$$

where v_w is a radial wall velocity.

Because within the scope of the second order nonlinear theory we neglect terms which are of the third order or higher, it is possible to simplify Eq. (9) by using relation (11) and adopting the supposition $v_w\sim\mu^2$,

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=-\frac{2\rho_0}{r_0}v_w. \quad (12)$$

Using the linear relation

$$\frac{\partial p}{\partial t}=-\rho_0\frac{\partial^2\varphi}{\partial t^2}, \quad (13)$$

we can write the following convolution integral for the radial wall velocity v_w:

$$v_w=\int_{-\infty}^{t}k_w(t-t')\frac{\partial p(x,t')}{\partial t'}dt' =-\rho_0\int_{-\infty}^{t}k_w(t-t')\frac{\partial^2\varphi(x,t')}{\partial t'^2}dt', \quad (14)$$

where $k_w(t)$ is a kernel function representing behavior of the considered tube wall.

If we take into account both the acoustic boundary layer [Eq. (6)] and the tube elasticity [Eq. (12)] together with relation (14) we obtain the resulting continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\frac{2\rho_0^2}{r_0}\int_{-\infty}^{t}k_w(t-t')\frac{\partial^2\varphi}{\partial t'^2}dt'+\frac{2\rho_0 B}{c_0^2}\frac{\partial^{3/2}\varphi}{\partial t^{3/2}}. \quad (15)$$

Using the basic equations of hydromechanics, namely, the Navier–Stokes equation of motion, the heat transfer equation, the state equation (see, e.g., Refs. 19–21), and the modified continuity equation (15) we can derive in the second approximation the following one-dimensional modified Kuznetsov's equation[22] for the velocity potential φ:

$$\frac{\partial^2\varphi}{\partial t^2}-c_0^2\frac{\partial^2\varphi}{\partial x^2}=\frac{b}{\rho_0 c_0^2}\frac{\partial^3\varphi}{\partial t^3}-\frac{\partial}{\partial t}\left[\frac{\beta-1}{c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\varphi}{\partial x}\right)^2\right] -\frac{2\rho_0 c_0^2}{r_0}\int_{-\infty}^{t}k_w(t-t')\frac{\partial^2\varphi}{\partial t'^2}dt'-2B\frac{\partial^{3/2}\varphi}{\partial t^{3/2}}, \quad (16)$$

where c_0 is the small-signal sound speed, $b=\zeta+4\eta/3+\varkappa(1/c_V-1/c_p)$ is the coefficient of sound diffusivity, ζ and η are the coefficients of bulk and shear viscosity, and β is the coefficient of nonlinearity.

As the profile of simple waves varies slowly in the space we can search for the solution of Eq. (16) in the moving reference frame in the form

$$\varphi=\varphi\left(x_1=\mu x,\tau=t-\frac{x}{c_0}\right). \quad (17)$$

Then taking into consideration that $b\sim B\sim\mu$ we can derive the following modified Burgers equation from Eq. (16) within frame of the second order nonlinear theory:

$$\frac{\partial v}{\partial x}-\frac{\beta}{c_0^2}v\frac{\partial v}{\partial\tau}+\frac{\rho_0 c_0}{r_0}\int_{-\infty}^{\tau}k_w(\tau-\tau')\frac{\partial v}{\partial\tau'}d\tau'+\frac{B}{c_0}\frac{\partial^{1/2}v}{\partial\tau^{1/2}} -\frac{b}{2\rho_0 c_0^3}\frac{\partial^2 v}{\partial\tau^2}=0. \quad (18)$$

It is obvious that it is possible to use the model [Eq. (18)] also for the nonlinear plane waves which propagate through a hard-walled tube. In this case we can consider the term representing the elastic properties of the tube wall as equal to zero.[15]

B. Derivation of dispersion relations

Under the above mentioned restrictions, the tube can be regarded as consisting simply of a series of ring-shaped elements whose radial motion is caused only by elastic circumferential stresses and radial inertia. As the tube wall yields locally to the inner pressure we can write the following expression for the force acting on the ring:

$$F(t)=p(t)A(t)=p(t)[A_0+A'(t)]=p(t)2\pi r(t)\Delta l =p(t)[2\pi r_0\Delta l+2\pi r'(t)\Delta l], \quad (19)$$

where p is an inner pressure and Δl is the width of the ring element. If we suppose that r' is sufficiently small compared to r_0 then we can simplify relation (19),

$$F(t)\simeq p(t)A_0. \quad (20)$$

With using the complex representation we can rewrite Eq. (20) as

$$\hat{F}\simeq\hat{p}A_0. \quad (21)$$

For the ring element we can use the equivalent electromechanic circuit which is sketched in Fig. 1. We suppose that air surrounds the ring and consequently the radiation load on the outside of the ring can be neglected.[23]

On the basis of the equivalent circuit we obtain

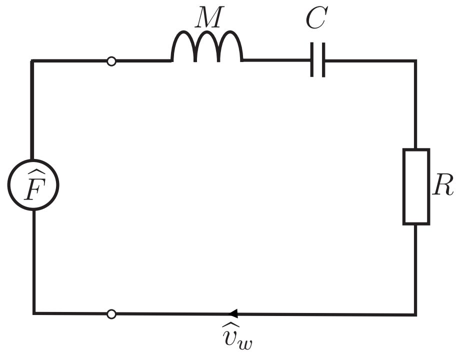

FIG. 1. The equivalent electromechanic circuit, M is the inertance, C is the compliance, R is mechanical resistance, and $\hat{v}_w$ is radial wall velocity.

$$\hat{F} = \left(R + j\omega M + \frac{1}{j\omega C}\right)\hat{v}_w = \hat{Z}_w \hat{v}_w, \tag{22}$$

where M is the inertance, C is the compliance, R is mechanical resistance, and $\hat{Z}_w$ is the resulting wall mechanical impedance.

Using relations (21) and (22) we can write

$$\hat{v}_w \simeq \frac{A_0}{\hat{Z}_w}\hat{p} = \frac{2\pi r_0 \Delta l}{\hat{Z}_w}\hat{p}. \tag{23}$$

With the help of the convolution integral (14) we can also write that

$$\hat{v}_w = j\omega \hat{k}_w \hat{p}. \tag{24}$$

By comparing Eqs. (23) and (24) we obtain

$$\hat{k}_w = \frac{2\pi r_0}{j\omega \hat{Z}_m}, \tag{25}$$

where

$$\hat{Z}_m = \frac{\hat{Z}_w}{\Delta l} = R_m + j\omega M_m + \frac{1}{j\omega C_m}, \tag{26}$$

where $R_m = R/\Delta l$ is the specific mechanical resistance (kg m^{-1} s^{-1}), $M_m = M/\Delta l$ is the specific inertance (kg m^{-1}), and $C_m = C\Delta l$ (kg^{-1} m s^2) is the specific compliance.

With help of expressions (25) and (26) it is possible to write (see Ref. 13)

$$\frac{2\hat{k}_w}{r_0} = \frac{4\pi}{j\omega \hat{Z}_m} = \frac{4\pi C_m}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + j\omega R_m C_m}, \tag{27}$$

where ω_m represents the mechanical resonance angular frequency which is given by the following relation:

$$\omega_m = \frac{1}{\sqrt{C_m M_m}}. \tag{28}$$

After linearizing modified Kuznetsov's equation

$$\frac{\partial^2 \varphi}{\partial t^2} - c_0^2 \frac{\partial^2 \varphi}{\partial x^2} - \frac{b}{\rho_0 c_0^2}\frac{\partial^3 \varphi}{\partial t^3} + \frac{2\rho_0 c_0^2}{r_0}\int_{-\infty}^{t} k_w(t-t')\frac{\partial^2 \varphi}{\partial t'^2}dt' + 2B\frac{\partial^{3/2}\varphi}{\partial t^{3/2}} = 0, \tag{29}$$

we can find the following dispersion relation:

$$k = \frac{\omega}{c_0}\sqrt{1 - j\frac{\omega b}{\rho_0 c_0^2} + \frac{2\rho_0 c_0^2 \hat{k}_w}{r_0} - \frac{2B(j\omega)^{3/2}}{\omega^2}}, \tag{30}$$

where k is the complex wave number.

Expression (30) can be simplified on the basis of the binomial series

$$k \simeq \frac{\omega}{c_0}\left(1 - j\frac{\omega b}{2\rho_0 c_0^2} + \frac{\rho_0 c_0^2 \hat{k}_w}{r_0} + \frac{B}{\sqrt{j\omega}}\right). \tag{31}$$

Since $\hat{k}_w / r_0$ is complex, see relation (27), we can write

$$\frac{\hat{k}_w}{r_0} = \Re\left(\frac{\hat{k}_w}{r_0}\right) + j\Im\left(\frac{\hat{k}_w}{r_0}\right) = \frac{2\pi C_m\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]}{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2 + R_m^2 C_m^2 \omega^2} - j\frac{2\pi R_m C_m^2 \omega}{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2 + R_m^2 C_m^2 \omega^2}. \tag{32}$$

It is possible to simplify formula (32) when the following relations are satisfied:[13]

$$\omega \ll \omega_m, \quad \omega R_m C_m \ll 1. \tag{33}$$

We can write that

$$\frac{\hat{k}_w}{r_0} \simeq 2\pi C_m\left[1 + \left(\frac{\omega}{\omega_m}\right)^2\right] - j2\pi R_m C_m^2 \omega\left[1 + 2\left(\frac{\omega}{\omega_m}\right)^2\right]. \tag{34}$$

Expression (34) is valid only for lower frequencies; it means that the dispersion and dissipation effects dominate above the nonlinear ones. In the opposite case the higher harmonic components, which arise in the course of the wave propagation, do not satisfy conditions (33). When we can take into account only small number of harmonics then it is further possible to simplify relation (34) as follows:

$$\frac{\hat{k}_w}{r_0} \simeq 2\pi C_m\left[1 + \left(\frac{\omega}{\omega_m}\right)^2\right] - j2\pi R_m C_m^2 \omega. \tag{35}$$

C. The KdVB and Kuramoto–Sivashinsky model equation for an elastic tube

We can rewrite Eq. (18) into the form[24]

$$\frac{\partial v}{\partial x} - \frac{\beta}{c_0^2}v\frac{\partial v}{\partial \tau} + \int_{-\infty}^{\infty} K(\tau - \tau')\frac{\partial v(x,\tau')}{\partial \tau'}d\tau' = 0, \tag{36}$$

where $K(\tau)$ represents a kernel function which describes assumed dispersion and dissipation properties. Suppose we know the dispersion relation

TABLE I. Real parameters for air and the used elastic tube $(23\ ^{\circ}\mathrm{C})$.

c_0 $(\mathrm{m\,s^{-1}})$	ρ_0 $(\mathrm{kg\,m^{-3}})$	b $(\mathrm{kg\,m^{-1}\,s^{-1}})$	B $(\mathrm{s^{-1/2}})$	C_m $(\mathrm{m\,s^2\,kg^{-1}})$	R_m $(\mathrm{kg\,m^{-1}\,s^{-1}})$	M_m $(\mathrm{kg\,m^{-1}})$
345.22	1.193	4.578×10^{-5}	0.7227	4.45×10^{-8}	154.0	0.118

$$D(\omega^*,k')=0, \tag{37}$$

which corresponds to the kernel function $K(\tau)$, $\omega^*=\omega$ and

$$k'=k-\omega/c_0=\frac{\omega}{c_0}\left(\frac{\rho_0c_0^2\hat{k}_w}{r_0}+\frac{B}{\sqrt{j\omega}}-j\frac{\omega b}{2\rho_0c_0^2}\right) \tag{38}$$

is a complex wave number that is related to the independent variables τ, x in contrast to the complex wave number (31) that is connected with the independent variables t and x. Provided that certain conditions, which are discussed later, are fulfilled then we can simplify the dispersion relation (37) into the asymptotic form

$$D_a(\omega,k')=0. \tag{39}$$

After linearizing Eq. (36) we can write[25]

$$\frac{\partial v}{\partial x}+\int_{-\infty}^{\infty}K_a(\tau-\tau')\frac{\partial v(x,\tau')}{\partial\tau'}d\tau'=0, \tag{40}$$

where the kernel function $K_a(\tau)$ corresponds to asymptotic dispersion relation (39). We assume the solution of Eq. (40) in the form $v=v_m\exp[j(\omega\tau-k'x)]$ (v_m is the acoustic velocity amplitude). We can substitute this solution into Eq. (40) and we obtain

$$-jk'+\int_{-\infty}^{\infty}K_a(\tau-\tau')j\omega\exp[-j\omega(\tau-\tau')]d\tau'=0. \tag{41}$$

Equation (41) can be rewritten into the form

$$\frac{k'}{\omega}=\mathcal{F}[K_a(\tau)]=\int_{-\infty}^{\infty}K_a(\tau)\exp(-j\omega\tau)d\tau, \tag{42}$$

where $\mathcal{F}[\cdot]$ represents the Fourier transform. The rhs of Eq. (42) is the Fourier transform of the given kernel $K_a(\tau)$. On the basis of the inverse Fourier transform we have

$$K_a(\tau)=\mathcal{F}^{-1}\left[\frac{k'}{\omega}\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k'}{\omega}\exp(j\omega\tau)d\omega. \tag{43}$$

Substituting expression (34) into relation (38) we obtain

$$\begin{aligned}k'\simeq\omega\Bigg\{&2\pi\rho_0c_0C_m\left[1+\left(\frac{\omega}{\omega_m}\right)^2\right]\\&-j2\pi\rho_0c_0R_mC_m^2\omega\left[1+2\left(\frac{\omega}{\omega_m}\right)^2\right]\\&+\frac{B}{c_0\sqrt{j\omega}}-j\frac{\omega b}{2\rho_0c_0^3}\Bigg\}.\end{aligned} \tag{44}$$

On the basis of relation (43) we can find the asymptotic kernel function $K_a(\tau)$ by using formula (44). After substitution of this asymptotic kernel function into Eq. (36) we get

$$\begin{aligned}&\frac{\partial v}{\partial x}-\frac{\beta}{c_0^2}\left(v-\frac{2\pi\rho_0c_0^3C_m}{\beta}\right)\frac{\partial v}{\partial\tau}+\frac{B}{c_0}\frac{\partial^{1/2}v}{\partial\tau^{1/2}}\\&\quad-\frac{b+4\pi\rho_0^2C_m^2R_mc_0^4}{2\rho_0c_0^3}\frac{\partial^2v}{\partial\tau^2}-\frac{2\pi\rho_0c_0R_mC_m}{\omega_m^2}\frac{\partial^3v}{\partial\tau^3}\\&\quad+\frac{4\pi\rho_0c_0R_mC_m^2}{\omega_m^2}\frac{\partial^4v}{\partial\tau^4}=0.\end{aligned} \tag{45}$$

Equation (45) represents the Kuramoto–Sivashinsky (Benney) equation[26,27] which is modified by the term representing the boundary layer effects. This equation can be simplified into the KdVB equation (see, e.g., Ref. 28) by setting the last term equal to zero when propagation wave distances are relatively short or the source frequency is sufficiently low and the dispersive and dissipative effects are insofar important that it is possible to take into account a reasonably small number of harmonics.

III. ANALYSIS OF NUMERICAL SOLUTIONS

This section is devoted to qualitative analysis of numerical results of the model equation (18) for a silicon rubber tube (hose). This analysis enables to show different scenarios of possible wave evolution for various source frequencies and hence demonstrates how the individual terms in the model equation depend on the source frequency. The real physical parameters are presented in Table I and for these parameters the analysis of numerical solutions is carried out. In order to show frequency limits of the KdVB and Kuramoto–Sivashinsky equation we can depict the courses of function $\hat{k}_w/r_0$ in dependence on frequency according to relations (32), (34), and (35), see Figs. 2 and 3. It is evident from Figs. 2 and 3 that in this case the Kuramoto–

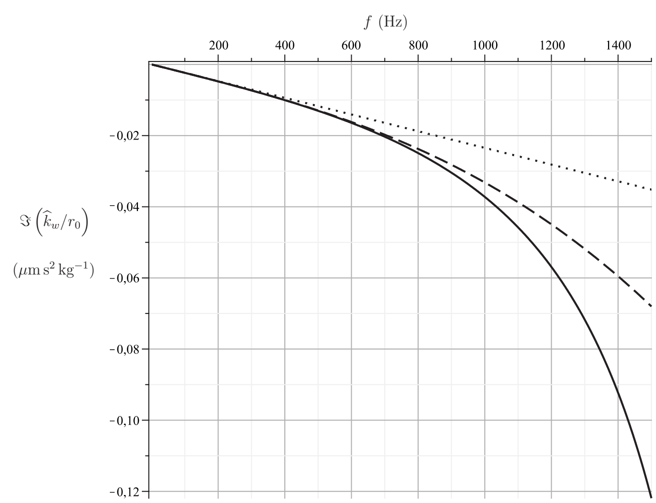

FIG. 2. Comparison of imaginary parts of relations (32) (solid line), (34) (dashed line), and (35) (dotted line).

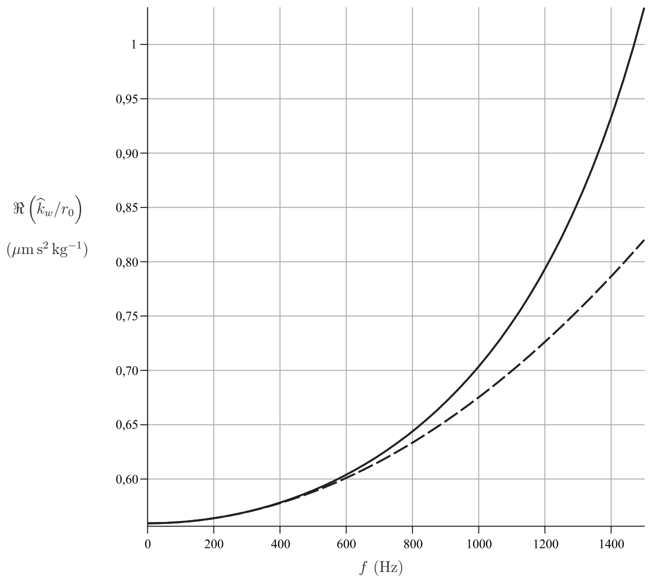

FIG. 3. Comparison of real parts of relations (32) (solid line) and (34) (dashed line).

Sivashinsky equation can be used in the frequency range approximately up to 1000 Hz, whereas the KdVB equation only up to about 800 Hz.

The model equation (18) takes into account three effects, i.e., the nonlinear, dissipative, and dispersion ones. There is a competition between the nonlinear effect, on the one hand, and the dispersion and dissipative effect on the other hand. In this case we can estimate a wave behavior on the basis of the characteristic lengths (see, e.g., Ref. 19). The nonlinear length is given as

$$L_N = \frac{c_0^2}{\beta v_m \omega}, \tag{46}$$

the dissipative length can be expressed as

$$L_D = \frac{1}{\alpha}, \tag{47}$$

and the dispersion or coherent length for the second harmonic component as

$$L_C = \frac{\pi}{\left| 2\omega \left[\dfrac{1}{c_{\mathrm{ph}}(2\omega)} - \dfrac{1}{c_{\mathrm{ph}}(\omega)} \right] \right|}. \tag{48}$$

It holds that the shorter characteristic length means the greater influence of the considered effect (in our case the diffraction length tends to infinity because we suppose that plane waves propagate in the tube). On the basis of the characteristic lengths we can outline several scenarios of possible wave evolutions with respect to a chosen source frequency (amplitude of the source is supposed to be constant, 4000 Pa).

First, we suppose that the source frequency is chosen so that $L_N/L_C > 1$ and $L_N/L_D < 1$. It means that the dispersion effects dominate the nonlinear ones and at the same time the nonlinear effects dominate the dissipative ones. For instance, the source frequency of 1000 Hz satisfies (if we suppose the

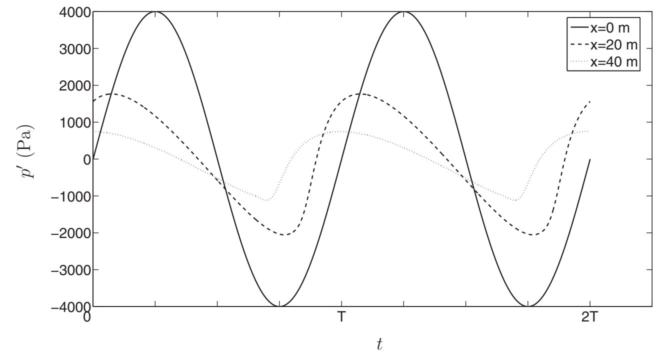

FIG. 4. Two periods of wave forms at different distances from the source; the source frequency is 80 Hz.

physical parameters in Table I) the above mentioned relations. For this case the wave distortion is negligible (the first harmonic component dominates) which affirms that dispersion effects dominate nonlinear ones and thus nonlinear acoustic interactions are ineffective. Thus, we could ignore the nonlinear effects and use only the linearized form of Eq. (18). When the supposed relations are satisfied also for lower source frequencies, in our case for a smaller source amplitude, it is possible to use the linearized Kuramoto–Sivashinsky or KdVB equation.

Further, we assume that the relations $L_N/L_C < 1$ and $L_N/L_D < 1$ are satisfied for at least the first ten harmonics. For this purpose we can choose the source frequency, e.g., 80 Hz. The satisfaction of these relations means that a sawtooth wave forms during propagation. The boundary layer dispersion causes the waveform asymmetry which occurs behind the shock formation. This result is demonstrated in Fig. 4. Furthermore, when the relations are satisfied then it is possible to replace Eq. (18) by the standard Burgers equation which is supplemented by the term representing the boundary layer effects.[15,16]

As the next case, we assume that the relations $L_N/L_C < 1$ and $L_N/L_D < 1$ are approximately satisfied only for the first two or three harmonics; hence, the generation of higher harmonics is suppressed. Figure 5 illustrates this situation.

When the boundary layer effects are small and again the relations $L_N/L_C < 1$ and $L_N/L_D < 1$ are satisfied, we can observe a gradual degeneration of the original waveform into individual solitons.[24,28,29] The solutions for this case are depicted in Figs. 6 and 7. In the figures we can see a train of the

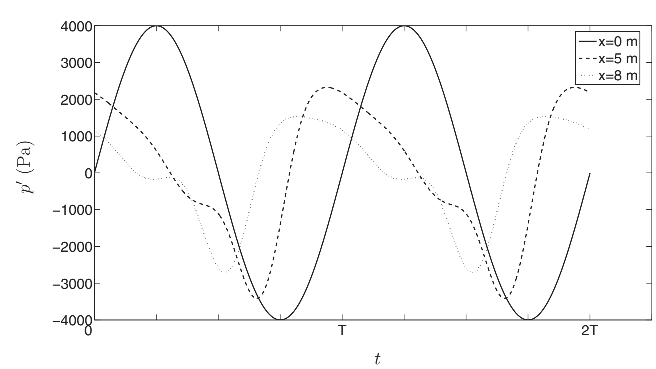

FIG. 5. Two periods of wave forms at different distances from the source; the source frequency is 400 Hz.

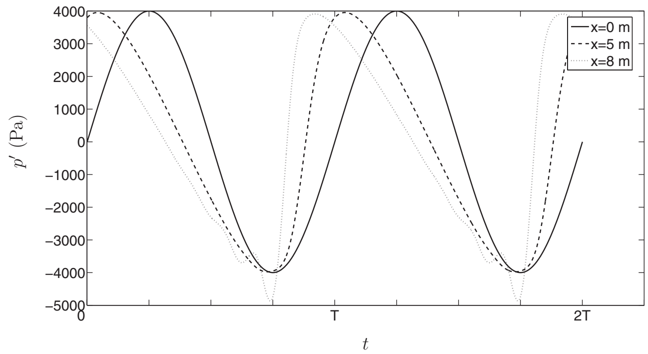

FIG. 6. Two periods of wave forms at different distances from the source; the source frequency is 200 Hz.

solitons which are gradually attenuated as the wave propagates through the considered tube. First, we can observe slight oscillations, see Fig. 6. This fact is related to the harmonics generation because dispersion effect starts to play a more important role for higher harmonics. Gradually the slight oscillations develop into a series of solitons which break up the previous waveform, see Fig. 7.

As the last case, we can consider the relations $L_N/L_C \ll 1$ and $L_N/L_D < 1$. It is obvious that $\hat{k}_w/r_0$, see formula (27), tends to zero for high source frequencies. It means that we can ignore the radial wall vibrations for high source frequencies. The considered relations for the characteristic lengths are satisfied for higher source frequencies.

IV. EXPERIMENT

To validate applicability of the model equations, in particular, the local reaction hypothesis, we compared the theoretical and experimental data.

A. Experimental setup

Traveling sound waves were driven in silicone-rubber hose (with inner diameter of 16 mm, wall thickness of 2 mm), length of the hose was 20 m, it was lengthened by 20 m of polyvinyl chloride hose of the same inner diameter in order to suppress standing waves due to reflections. All measurements were made at room temperature of 23 °C.

Harmonic driving signal was generated with direct-digital-synthesis function generator Motech FG 503 which was amplified by power amplifier Mackie M1400. High amplitude acoustic waves were generated by two speakers B&C 10MD26 (each of 350 W input) that were screwed one against another with a plastic ring for fastening of the hose. At the place where the hose was fastened to the plastic ring, reference 1/8 in. GRAS Type 40DP microphone was attached to measure the input acoustic pressure. In the hose, there were 13 small cuts distributed with 1 m distances for measurement of acoustic pressure with probe-microphone GRAS Type 40SA with stainless-steel 20 mm probe tube of 1.25 mm outer diameter (inner diameter of 1 mm).

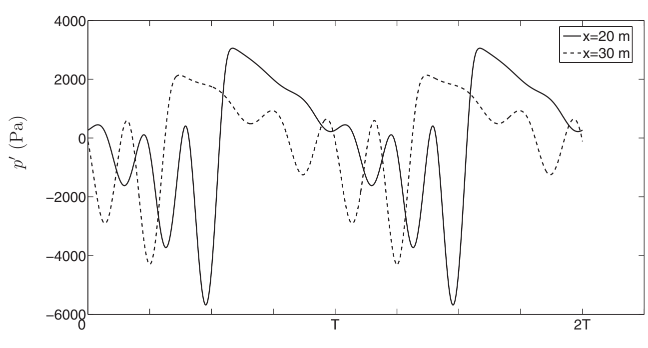

FIG. 7. Two periods of two wave forms at different distances from the source; the source frequency is 200 Hz.

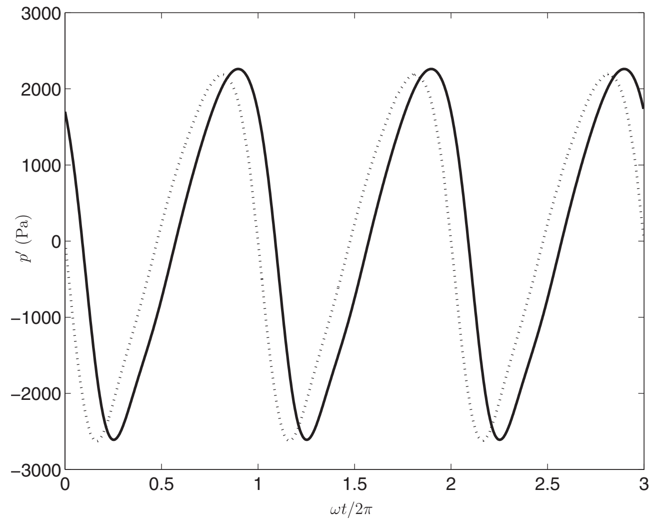

FIG. 8. Comparison of theoretical (solid line) and measured (dotted line) wave forms at the distance $x=3$ m, $p_m=3602$ Pa; the driving frequency is 400 Hz.

Measured signals were sampled by 16 bit National Instruments PCI-6251 plug-in computer data acquisition card at the rate of 100 kS/s per channel. Software for data acquisition and its processing was written in LABVIEW system.

On the basis of measurement of the silicone-rubber hose we found its material parameters which are shown in Table I.

In the course of measuring it was necessary to take into account the fact that the driving speakers heated up excessively. For this reason it was not possible to realize measuring with the same driving pressure at all points; hence the driving pressures were registered for individual measurements apart. The value of the driving pressure was around 3600 Pa ($v_m=8.74$ m s^{-1}). We confined ourselves to the relatively low source frequency in order that the dispersion and nonlinear effects could dominate above the dissipative ones.

B. Comparison of theoretical and experimental results

The measured and theoretical (computed numerically) data are depicted in Figs. 8–10. By comparison of measured and theoretical waveforms at different distances from the harmonic source it is clear that the model equation (18) enables to describe behavior of nonlinear traveling waves in the elastic tube and that the local reaction hypothesis is sufficient when the conditions mentioned in Sec. II are fulfilled. From these figures it is obvious that the shock waveform did not arise. This result is given by the fact that the relations $L_N/L_C < 1$ and $L_N/L_D < 1$ are fulfilled approximately only

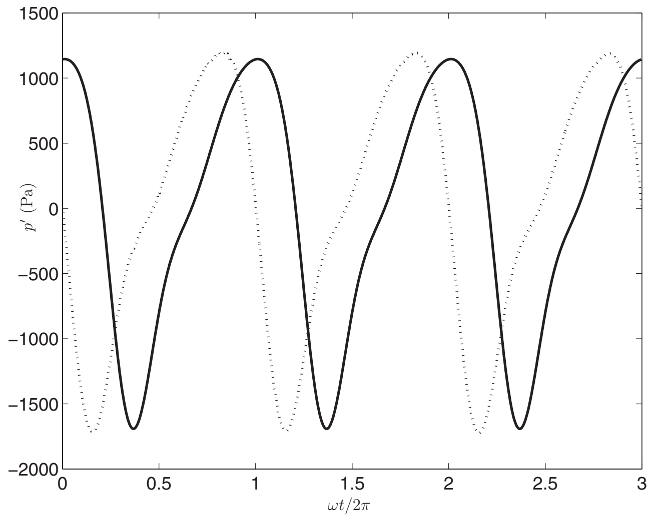

FIG. 9. Comparison of theoretical (solid line) and measured (dotted line) wave forms at the distance $x=7$ m, $p_m=3604$ Pa; the driving frequency is 400 Hz.

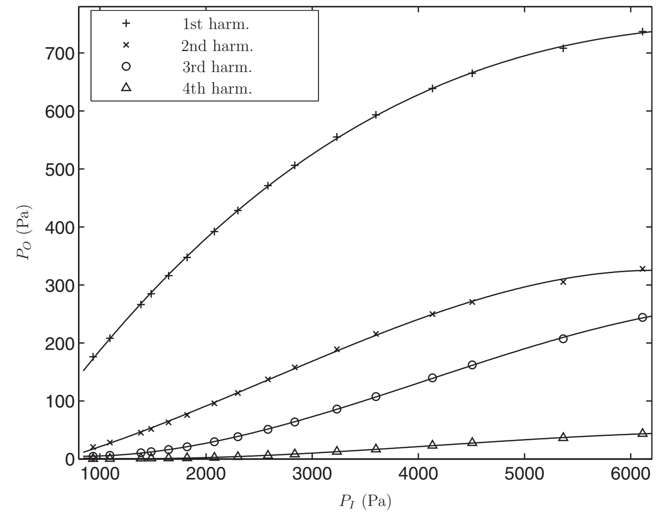

FIG. 11. Dependence of amplitude of the first four harmonics at distance of 13 m on input pressure amplitude.

for the first three harmonics; thus the generation of higher harmonics is suppressed. It means that the relation $L_N/L_C > 1$ holds for higher harmonics.

Figure 11 shows development of the first four harmonics of acoustic pressure at the distance of 13 m when input acoustic pressure ($f=333$ Hz) increases. It is apparent from the figure that even if dispersion is present, several higher harmonics appear and the amplitude of the first and second harmonics is not proportional to the pressure amplitude at the input. This effect is called *acoustic saturation* and it has connection with the nonlinear attenuation. It is obvious from Fig. 11 that the fourth harmonics and the third one depend on the pressure amplitude at the input almost linearly. It means that the cascade process of harmonics generation is interrupted or in other words, the characteristic coherent length is shorter than the nonlinear one ($L_N > L_C$) for higher harmonics in contrast to the lower ones.

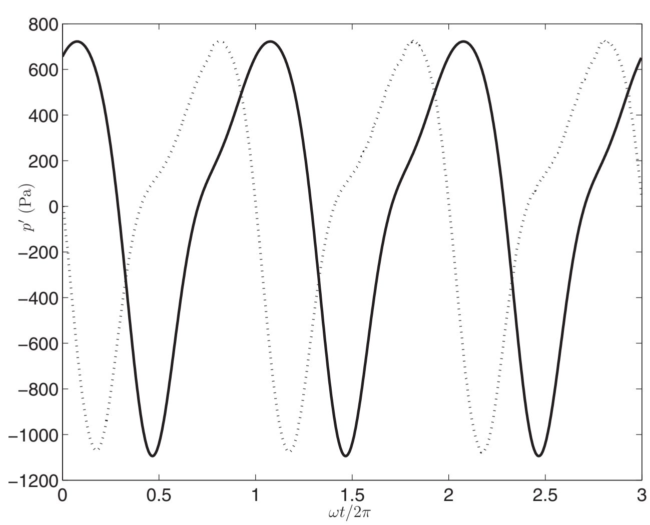

FIG. 10. Comparison of theoretical (solid line) and measured (dotted line) wave forms at the distance $x=10$ m, $p_m=3597$ Pa; the driving frequency is 400 Hz.

V. CONCLUSION

We derived the modified Burgers equation (18) which enables to describe nonlinear acoustic plane waves propagating through the elastic tube filled with a thermoviscous gas. The model equation takes into account only the breathing circumferential damped vibrations of the considered tube wall. Since the viscosity of gas is considered, it is necessary to suppose that the thin acoustic boundary layer appears near the tube wall. The acoustic boundary layer affects the acoustic field outside of this layer (the mainstream) in a considerable way. For this reason the derived model equation also contains the term which represents the boundary layer effects. The applicability of the modified Burgers equation was verified experimentally. The realized measurements demonstrate the fact that the dispersion induced by the elastic tube wall causes nonlinear acoustic interactions which are almost ineffective and further that it is not possible to ignore the influences of the acoustic boundary layer. It means that the shock formation does not arise and so we can observe only a small number of harmonics. Because we consider nonlinear, dispersion, and dissipative wave effects, that compete each other, it is possible to distinguish different regimes. These regimes are classified by means of the relations between characteristic lengths of the supposed wave effects. The supposed regimes lead to different wave evolutions. In the case when the nonlinear effects dominate the dispersion and dissipative one then the original waveform degenerates gradually into individual solitons. However, we can observe the solitons only when the acoustic boundary layer effects are weak. Contrary if it is not possible to suppose that these effects are weak (e.g., in the case of a small tube radius) then the generation of solitons is suppressed. This fact shows how the boundary layer plays important role.

Using the long-wave approximation, the modified Burgers equation was reduced to the KdVB and Kuramoto–Sivashinsky equations, which were investigated by many authors; however, they have not studied influence of the boundary layer effects. Considering that acoustic boundary layer plays more important role for lower frequencies, we cannot neglect its influence in the case of long-wave approximation.

Considering that the KdVB equation can be used only for a relatively narrow frequency range we derived the Kuramoto–Sivashinsky equation which can be used for a wider frequency range and thus it can take into account more higher harmonics correctly.

We intend to extend further the presented results and methods for the case of nonlinear standing waves in elastic resonators.

ACKNOWLEDGMENT

This work was supported by GACR Grant No. 202/09/1509.

APPENDIX: FRACTIONAL DERIVATIVE OPERATOR

Let us introduce the following fractional integration operator of the order α:

$$I^{\alpha}[f(t)] = \int_{-\infty}^{t} \frac{(t-\chi)^{\alpha-1}}{\Gamma(\alpha)} f(\chi) d\chi, \tag{A1}$$

where Γ is the gamma function. The αth fractional derivative is given as

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = I^{n-\alpha}\left[\frac{d^{n} f(t)}{dt^{n}}\right], \tag{A2}$$

where $n=1+[\alpha]$, where $[\alpha]$ represents the whole part of α.

On the basis of the mentioned definition of the fractional derivative we can write for the fractional derivative of the order $\alpha=3/2$ that

$$\frac{d^{3/2} f(t)}{dt^{3/2}} = I^{2-3/2}\left[\frac{d^{2} f(t)}{dt^{2}}\right] = I^{1/2}\left[\frac{d^{2} f(t)}{dt^{2}}\right] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{d^{2} f(\chi)}{d\chi^{2}} \frac{d\chi}{\sqrt{t-\chi}}, \tag{A3}$$

where we used that $\Gamma(1/2)=\sqrt{\pi}$.

[1] R. D. Fay, R. L. Brown, and O. V. Fortier, "Measurement of acoustic impedances of surfaces in water," J. Acoust. Soc. Am. **19**, 850–856 (1947).
[2] W. J. Jacobi, "Propagation of sound waves along liquid cylinders," J. Acoust. Soc. Am. **21**, 120–127 (1949).
[3] G. W. Morgan and J. P. Kiely, "Wave propagation in a viscous liquid contained in flexible tube," J. Acoust. Soc. Am. **26**, 323–328 (1954).
[4] M. M. Sondhi, "Model for wave propagation in a lossy vocal tract," J. Acoust. Soc. Am. **55**, 1070–1075 (1974).
[5] R. W. Guelke and A. E. Bunn, "Transmission line theory applied to sound wave propagation in tubes with compliant walls," Acustica **48**, 101–106 (1981).
[6] J. J. Fredberg, M. E. B. Wohl, G. M. Glass, and H. L. Dorkin, "Airway area by acoustic reflections measured at the mouth," J. Appl. Physiol. **48**, 749–758 (1980).
[7] L. Elvira-Segura, "Acoustic wave dispersion in a cylindrical elastic tube filled with a viscous liquid," Ultrasonics **37**, 537–547 (2000).
[8] R. Pico and F. Gautier, "The vibroacoustics of slightly distorted cylindrical shells: A model of the acoustic input impedance," J. Sound Vib. **302**, 18–38 (2007).
[9] F. Gautier, J. Gilbert, J.-P. Dalmont, and R. P. Vila, "Wave propagation in a fluid filled rubber tube: Theoretical and experimental results for Korteweg's wave," Acta. Acust. Acust. **93**, 333–344 (2007).
[10] V. A. D. Grosso, "Analysis of multimode acoustic propagation in liquid cylinders with realistic boundary conditions—Application to sound speed and absorption measurements," Acustica **24**, 299–311 (1971).
[11] S. Yomosa, "Solitary waves in large blood vessels," J. Phys. Soc. Jpn. **56**, 506–520 (1987).
[12] S. E. H. A. Erbay and S. Dost, "Wave propagation in fluid filled nonlinear viscoelastic tubes," Acta Mech. **95**, 87–102 (1992).
[13] T. Kamakura and Y. Kumamoto, "Waveform distortions of finite amplitude acoustic wave in an elastic tube," in Frontiers of Nonlinear Acoustics: Proceedings of the 12th ISNA, edited by M. F. Hamilton and D. T. Blackstock (1991), pp. 333–338.
[14] W. Chester, "Resonant oscillations in closed tubes," J. Fluid Mech. **18**, 44–64 (1964).
[15] S. N. Makarov and E. V. Vatrushina, "Effect of the acoustic boundary layer on a nonlinear quasiplane wave in a rigid-walled tube," J. Acoust. Soc. Am. **94**, 1076–1083 (1993).
[16] M. Bednarik and P. Konicek, "Propagation of quasiplane nonlinear waves in tubes and the approximate solutions of the generalized Burgers equation," J. Acoust. Soc. Am. **112**, 91–98 (2002).
[17] I. M. Sokolov, J. Klafter, and A. Blumen, "Fractional kinetics," Phys. Today **55**, 48–54 (2002).
[18] M. C. Junger and D. Feit, *Sound, Structures, and Their Interaction* (MIT Press, Cambridge, 1986).
[19] O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Consultants Bureau, New York, 1977).
[20] K. Naugolnykh and L. Ostrovsky, *Nonlinear Wave Processes in Acoustics* (Cambridge University Press, Cambridge, 1998).
[21] M. F. Hamilton and D. T. Blackstock, *Nonlinear Acoustics* (Academic, New York, 1997).
[22] S. Makarov and M. Ochmann, "Nonlinear and thermoviscous phenomena in acoustics, Part II," Acta. Acust. Acust. **83**, 197–222 (1997).
[23] M. L. Munjal and P. T. Thawani, "Acoustic analysis and design of compliant cable-hose systems," Noise Control Eng. J. **45**, 235–242 (1997).
[24] J. Engelbrecht, *Nonlinear Wave Dynamics* (Kluwer Academic, Dordrecht, 1997).
[25] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1999).
[26] A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations* (Chapman and Hall, London, 2004).
[27] L. Jiang, X. Chen, Z.-T. Fu, S.-K. Liu, and S.-D. Liu, "Periodic solutions to KdV-Burgers-Kuramoto equation," Commun. Theor. Phys. **45**, 815–818 (2006).
[28] V. R. Kodali, "Spectral analysis of numerical solutions to the Burgers-Korteweg-DeVries equation," J. Acoust. Soc. Am. **79**, 26–30 (1986).
[29] N. J. Zabusky and M. D. Kruskal, "Interactions of solitons in a collisionless plasma and the recurrence of initial states," Phys. Rev. Lett. **15**, 240–243 (1965).