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Propagation of quasiplane nonlinear waves in tubes and the approximate solutions of the generalized Burgers equation

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This paper deals with using the generalized Burgers equation for description of nonlinear waves in circular ducts. Two new approximate solutions of the generalized Burgers equation (GBE) are presented. These solutions take into account the boundary layer effects. The first solution is valid for the preshock region and gives more precise results than the Fubini solution, whereas the second one is valid for the postshock (sawtooth) region and provides better results than the Fay solution. The approximate solutions are compared with numerical results of the GBE. Furthermore, the limits of validity of the used model equation are discussed with respect to boundary conditions and radius of a circular duct. © 2002 Acoustical Society of America. [DOI: 10.1121/1.1488940]

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I. INTRODUCTION

Propagation of nonlinear sound waves in waveguides represents a very interesting physical problem. In the case that nonlinear waves travel in a gas-filled waveguide, we can observe phenomena such as nonlinear distortion, nonlinear absorption, diffraction, lateral dispersion, boundary layer effects, etc. All these phenomena can be described by means of the complete system of the equations of the hydrodynamics, see, e.g., Refs. 1–4: the Navier–Stokes momentum equation, the continuity (mass conservation) equation, the heat transfer (entropy) equation and the state equations. Unfortunately, we have not known a general solution of this system of equations and numerical solutions bring many problems regarding stability of the solutions and their time consumption. Consequently, it is sensible to simplify the fundamental system of equations if we ignore some phenomena or consider some of them weak. This simplification leads to a derivation of model equations of nonlinear acoustics.

There are a number of papers which are devoted to various aspects concerning propagation of nonlinear waves in waveguides. The viscous and thermal dissipative effects on the nonlinear propagation of plane waves in hard-walled ducts are treated for instance in the papers.5–7 The authors deal with the dependence of the frequency on the dissipative and dispersive effects induced by the acoustic boundary layer. Experimental results focused on propagation of finite-amplitude plane waves in circular ducts are presented in Refs. 8 and 9. Here is demonstrated a very good agreement between experimental data and results obtained by means of the Rudnick decay model for the fundamental harmonic. Burns10 obtained a fourth-order perturbation solution for finite-amplitude waves. However, his expansion breaks down for large times because it contains secular terms. He took into account dissipation but he neglected the mainstream dissipation with respect to the boundary dissipation. Keller and Millmann11 found the solution of the model equation for inviscid isentropic fluids where they used a perturbation expansion adapted to eliminate secular terms and determined the nonlinear wave number shift for dispersive modes. Keller12 utilized the results from Ref. 11. He rewrote the results in a form that is useful near the cutoff frequency, in order to show that the cutoff frequencies and resonant frequencies of modes in acoustic waveguides of finite length depend upon the mode amplitude. Nayfeh along with Tsai13 presented the nonlinear effects of the gas motion as well as the lining nonlinear acoustic material properties on the wave propagation and attenuation in circular ducts. They obtained a second-order uniformly valid expansion by using the method of multiple scales. These authors presented work,14 where they investigated nonlinear propagation in a rectangular duct whose side walls were acoustically treated by means of the method of multiple scales as well. Also Ginsberg15 dealt with the nonlinear propagation in the rectangular ducts. He determined by an asymptotic method the nonlinear two-dimensional acoustic waves that occur within a rectangular duct of semi-infinite length as the result of periodic excitation. In the work16 he utilized the perturbation method of renormalization to study the effect nonlinearity on a hard-walled rectangular waveguide. Nonlinear wave interaction in a rectangular duct was investigated by Hamilton and TenCate in Refs. 17 and 18 as well. Multiharmonic excitation of a hard-walled circular duct was treated by Nayfeh in the work.19 He used the method of multiple scales to derive a nonlinear Schrödinger equation for the temporal and spatial modulation of the amplitudes and the phases of waves propagating in a hard-walled duct. Foda presented his work in Ref. 20 which is concerned with the nonlinear interactions and propagation of two primary waves in higher order modes of a circular duct each at an arbitrary different frequency and finite amplitude. He used the renormalization method to annihilate secular terms in the obtained expression. If we take into account no diffraction effects, we can use for description of nonlinear plane waves in circular ducts the generalized Burgers equation. It is the Burgers equation which is supplemented by the term which represents boundary layer effects, see Refs. 21–25. The generalized Burgers equation enables to describe dissipative and dispersive effects that are caused by the boundary layer. Asymptotical and numerical solutions of this equation were presented by Sugimoto.26 The approximate solution of the Burgers equation in the preshock region
II. GENERALIZED BURGERS EQUATION

If we assume the wall friction in the case that nonlinear waves propagate in a rigid tube, then a thin boundary layer appears near the tube wall. Within the boundary layer, the velocity component in the direction of the tube axis decreases from a mainstream value to zero at the tube wall. The boundary layer affects the acoustic waves not only near the walls but even in the entire volume. The boundary layer causes both energy dissipation and wave dispersion.

According to the second-order nonlinear theory we can obtain the generalized Burgers equation (GBE) which enables to describe weakly nonlinear waves in thermoviscous fluids when the boundary layer is assumed to have a small displacement effect on the mainstream

\[
\frac{\partial \nu}{\partial z} - \frac{\beta}{c_0} \frac{\partial \nu}{\partial \tau} + \sqrt{2B} \frac{\partial^2 \nu}{\partial \tau^2} = \frac{b}{2} \frac{\partial^2 \nu}{\partial \tau},
\]

(1)

where \(\nu\) is the particle velocity, \(z\) is the coordinate along the axis of the tube, \(\tau = t - z/c_0\) is retarded time, \(t\) is time, \(c_0\) is the small-signal sound speed, \(b\) is the dissipation coefficient of the medium, \(\rho_0\) is the ambient density, \(\gamma\) is the adiabatic index, equal to the ratio of the specific heat at constant pressure \(c_p\) to that at constant volume \(c_v\), \(\beta = (\gamma + 1)/2\) is the coefficient of nonlinearity. The coefficient \(B\) corresponds with the boundary layer and is given as

\[
B = \sqrt{\frac{\nu}{2c_v^2G_0}} \left(1 + \frac{\beta c_v^2}{c_p \sqrt{\Pr}}\right),
\]

(2)

where \(R_0\) is the tube radius, \(\nu\) is the kinematic viscosity, \(\Pr\) is the Prandtl number, and \(\beta_T\) is the thermal expansion coefficient of fluid

\[
\beta_T = -\frac{1}{\rho_0} \left(\frac{\partial \rho}{\partial T}\right)_{p,T=T_0},
\]

(3)

where \(T\) is temperature. If we suppose a perfect gas then

\[
\beta_T c_p^2 = \gamma - 1.
\]

(4)

The fractional derivative of the order 1/2 in Eq. (1) represents the following integrodifferential operator:

\[
\frac{\partial^{1/2} \nu(z, \tau)}{\partial \tau^{1/2}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\tau} \frac{\partial \nu(z, \tau')}{\partial \tau'} \sqrt{\tau - \tau'} \, d\tau'.
\]

(5)

The third term in the left part of Eq. (1) represents the boundary layer effects and is valid on the condition that

\[
\delta \ll R_0, \quad \delta \ll \lambda,
\]

(6)

where \(\delta\) is the boundary layer thickness

\[
\delta = \sqrt{\frac{\nu}{\omega}}
\]

(7)

where \(\omega\) is an angular frequency.

The solution of the linearized GBE for the boundary condition \(\nu(0, \tau) = v_m \sin(\omega \tau)\) can be expressed as

\[
\nu = v_m \exp[-(\alpha + \alpha_b)z] \sin(\omega \tau - \alpha_b z),
\]

(8)

where \(\alpha\) is the attenuation coefficient for the classical thermoviscous loss mechanism

\[
\alpha = \frac{b \omega^2}{2 \rho_0 c_0}
\]

(9)

and the attenuation coefficient \(\alpha_b\) represents the losses due to the wall friction

\[
\alpha_b = B \sqrt{\omega}.
\]

(10)

The ratio of both attenuation coefficients can be written as

\[
\frac{\alpha}{\alpha_b} = \frac{\delta R_0}{\lambda}.
\]

(11)

It is obvious that \(\delta \lambda > 1\) for high frequencies but \(R_0/\lambda \gg 1\) because the first of the conditions (6) has to be satisfied. It means that the classical thermoviscous loss mechanisms for high frequencies in comparison to the boundary layer losses. Consequently, the condition \(\delta \lambda \ll 1\) cannot be satisfied for higher wave form harmonics.

III. APPROXIMATION SOLUTIONS OF THE GBE

We can write Eq. (1) in the following nondimensional form:

\[
\frac{\partial V}{\partial \sigma} - \frac{1}{2} \frac{\partial V^2}{\partial \theta} + \sqrt{2D_0} \frac{\partial^{1/2} V}{\partial \theta^{1/2}} = \frac{1}{G_0} \frac{\partial^2 V}{\partial \theta^2},
\]

(12)

where

\[
\theta = \omega \tau, \quad \sigma = \frac{\beta \nu_m \omega}{c_v^2 z}, \quad V = \frac{v}{v_m}, \quad \omega = \frac{b \omega}{\sqrt{\omega}}, \quad D_0 = \frac{B e_0 \sqrt{\omega}}{\omega v_m \beta}, \quad G_0 = \frac{2 \beta \nu_m \rho_0 c_0}{b \omega}.
\]

A. Preshock region

If we suppose the boundary condition

\[
V(0, \theta) = \sin(\theta),
\]

(14)

then we can seek the solution of Eq. (12) in the preshock region \(0 < \sigma < 1\) in the following form, see Ref. 29:

\[
V(\sigma, \theta) = A(\sigma) R(\sigma, \theta) = A(\sigma) \sin[\theta + \sigma A(\sigma) R(\sigma, \theta)],
\]

(15)
Using Eq. (15) we obtain
\[
\frac{\partial V}{\partial \sigma} = \frac{\partial (AR)}{\partial \sigma} + R \frac{\partial A}{\partial \sigma} + \frac{\cos(\theta + \sigma AR)\left(A + \sigma \frac{\partial A}{\partial \sigma}\right)AR}{1 - \sigma A \cos(\theta + \sigma AR)},
\]

where
\[
\frac{\partial V}{\partial \sigma} = \frac{\partial (AR)}{\partial \sigma} + A \cos(\theta + \sigma AR)\left(1 + \sigma \frac{\partial A}{\partial \sigma}\right)\frac{\partial R}{\partial \theta} - \frac{1}{1 - \sigma A \cos(\theta + \sigma AR)},
\]

\[
\frac{1}{G_0} \frac{\partial^2 V}{\partial \theta^2} = -\frac{A}{G_0} \sin(\theta + \sigma AR)\left(\frac{\partial R}{\partial \theta}\right)^2 + \frac{\sigma A^2}{G_0} \frac{\partial^2 R}{\partial \theta^2} \cos(\theta + \sigma AR).
\]

Provided that $1/G_0 \sim D_0 \sim \mu$ we can suppose $\sigma A \ll 1$. If we ignore terms which are higher than the second-order terms, then we can simplify Eqs. (16), (17), and (18)
\[
\frac{\partial V}{\partial \sigma} = R \frac{\partial A}{\partial \sigma} + \frac{A^2 R \cos(\theta + \sigma AR)}{1 - \sigma A \cos(\theta + \sigma AR)},
\]

\[
\frac{1}{G_0} \frac{\partial^2 V}{\partial \theta^2} = -\frac{1}{G_0} V
\]

or
\[
\frac{\partial V}{\partial \theta} = -\int V \, d\theta.
\]

If we integrate Eqs. (20) and (21), we suppose that the integration constants are equal to zero with respect to the boundary condition (14). With the help of the expressions (19), (20), and (21), we can modify Eq. (12),
\[
\frac{V}{A} \frac{dA}{d\sigma} + \frac{1}{G_0} V = -\sqrt{2} D_0 \frac{\partial^{1/2} V}{\partial \theta^{1/2}}.
\]

We can seek the approximate solution of Eq. (12) in the form of the modified Fubini solution, valid only in the pre-shock region $0 < \sigma < 1$,
\[
V = \text{Im} \left\{ \sum_{n=1}^{\infty} \frac{2J_n(n\sigma A_n(\sigma))}{n\sigma} \exp[j(n\theta + \Phi_n(\sigma))] \right\},
\]

where $j = \sqrt{-1}$ and $A_n(\sigma)$ is the function $A(\sigma)$ for the given harmonic because Eq. (22) is a linear equation, thus it is valid for each harmonic separately. We can assume that dispersion effects are very small in this region, therefore we can describe them by means of a function $\Phi_n(\sigma)$. This function can be expressed on the basis of the solution for the linearized GBE (8),
\[
\Phi_n(\sigma) = -D_0 \sqrt{n} \sigma.
\]

With the help of Eq. (21), we can express Eq. (5) as
\[
\frac{\partial^{1/2} V}{\partial \theta^{1/2}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\partial V(\sigma, \theta')}{\partial \theta'} \sqrt{\theta - \theta'} \, d\theta'.
\]

Substituting (23) into Eq. (25) we get
\[
\frac{\partial^{1/2} V}{\partial \theta^{1/2}} = -\text{Im} \left\{ \sum_{n=1}^{\infty} \frac{2J_n(n\sigma A_n(\sigma))}{n\sigma} \exp[j\Phi_n(\sigma)] \right\} \times \int_{-\infty}^{\infty} \frac{\exp(jn\theta')}{\sqrt{n}} \, d\theta' \right\}.
\]

We can solve the integral in Eq. (26),
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(jn\theta')}{\sqrt{\theta - \theta'}} \, d\theta' = -\frac{1}{\sqrt{2\pi}} \exp(jn\theta).
\]

Substituting Eq. (23) into Eq. (22) we obtain, with the help of expression (27), this equation for the function $A_n(\sigma)$
\[
\frac{dA_n}{d\sigma} + A_n \left( \frac{1}{G_0} + \frac{1}{\sqrt{n^2 + D_0}} \right) = 0.
\]

Here dispersion effects are not taken into account because these effects are incorporated in Eq. (23), see Ref. 29. Solving Eq. (28) we get
\[
A_n(\sigma) = C \exp \left[ -\sigma \left( \frac{1}{G_0} + n^{-3/2} D_0 \right) \right],
\]

where $C$ is an integration constant. For the case that $G_0 \to \infty$ and $D_0 \to 0$ the solution (23) has to represent the Fubini solution therefore the integration constant $C = 1$.

Using (23), (24), and (29) we can write the approximate solution of Eq. (12) in the region $0 < \sigma < 1$ in this form
\[
V(\sigma, \theta) = \text{Im} \left\{ \sum_{n=1}^{\infty} a_n \exp[j(n\theta - D_0 \sqrt{n} \sigma)] \right\},
\]

where
\[
a_n = \frac{2J_n(n\sigma \exp[-\sigma \left( \frac{1}{G_0} + n^{-3/2} D_0 \right)])}{n\sigma}.
\]

B. Postshock region

If we take into account only the boundary layer effects then Eq. (12) can be written as
\[
\frac{\partial V}{\partial \sigma} - \frac{1}{2} \frac{\partial V^2}{\partial \theta^2} + \sqrt{2} D_0 \frac{\partial^{3/2} V}{\partial \theta^{3/2}} = 0.
\]

We can seek the solution of Eq. (32) in the postshock region $\sigma \geq 3$ (after formation of a shock wave profile) in the following form, see Ref. 29:
evaluate the complex Fourier coefficients

\[
V(\sigma, \theta) = \frac{A(\sigma)(\pi - \theta)}{1 + \sigma A(\sigma)} = A(\sigma)R(\sigma, \theta)
\]

for \( \theta \in (0; 2\pi) \). (33)

The solution (33) applied to Eq. (32) yields

\[
\frac{\partial}{\partial \sigma}(AR) - A^2 R \frac{\partial R}{\partial \theta} = -\sqrt{2} D_0 \frac{\partial^{1/2} V}{\partial \theta^{1/2}}.
\]

After differentiating in Eq. (34) we obtain

\[
V \frac{dA_n}{d\sigma} - \sigma(dA_n/d\sigma) \frac{dA_n}{d\sigma} - \frac{A_n}{1 + \sigma A_n} V = -\sqrt{2} D_0 \frac{\partial^{1/2} V}{\partial \theta^{1/2}}.
\]

(35)

Fourier analysis of the solution (33) allows us to follow the behavior of the individual harmonics

\[
V(\sigma, \theta) = \text{Im} \left( \sum_{n=1}^{\infty} \frac{2}{n} \frac{A_n(\sigma)}{1 + \sigma A_n(\sigma)} \exp(jn\theta) \right),
\]

where \( A_n(\sigma) \) is the function \( A(\sigma) \) for the given harmonic because Eq. (35) is a linear equation with respect to the function \( V \), thus it is valid for each harmonic separately.

Substituting the spectral solution (36) in Eq. (35), we have

\[
\sum_{n=1}^{\infty} \frac{2}{n} \frac{A_n(\sigma)}{1 + \sigma A_n(\sigma)} \exp(jn\theta) \left[ \frac{\sigma dA_n}{d\sigma} + \frac{\partial^{1/2} V}{\partial \theta^{1/2}} (\frac{2}{n} p_n) \right] = 0,
\]

(37)

where

\[
p_n = \frac{A_n \exp(jn\theta)}{1 + \sigma A_n}.
\]

From Eq. (37) we can get the equations that enable us to evaluate the complex Fourier coefficients

\[
\frac{1}{A_n} \frac{dA_n}{d\sigma} - \sigma(dA_n/d\sigma) + \sqrt{2} j n D_0 = 0.
\]

(39)

After rewriting Eq. (39), we have

\[
\frac{dA_n}{d\sigma} + \sqrt{2} j n D_0 \sigma A_n^2 + \sqrt{2} j n D_0 A_n = 0,
\]

(40)

where Eq. (40) represents the Bernoulli differential equation. If we want to corporate dispersion effects, it is necessary to regard a function \( A_n \) as a complex function.

The general solution of Eq. (40) can be expressed in the following form:

\[
A_n = \frac{\exp(-\sigma \sqrt{2} j n D_0)}{C - \exp(-\sigma \sqrt{2} j n D_0) \left( 1 + \frac{1}{\sigma \sqrt{2} j n D_0} \right)},
\]

(41)

where \( C \) is an integration constant. It is necessary to choose the constant \( C \) in order to the coefficient \( A_n = 1 \) for the case when \( D_0 \to 0 \). That is,

\[
C = 1 + \frac{1}{\sqrt{2} j n D_0}.
\]

The constant \( C \) chosen by this way enables to reduce the solution (36) to the well-known solution for ideal fluids.

Substitution of the constant (42) into the solution (41) yields

\[
A_n = \frac{\exp(-\sigma \sqrt{2}j n D_0)}{1 - \exp(-\sigma \sqrt{2}j n D_0) \left( \frac{\exp(-\sigma \sqrt{2}j n D_0) - 1}{\sigma \sqrt{2}j n D_0} \right)}.
\]

(43)

Let us consider the Burgers equation

\[
\frac{\partial V}{\partial \sigma} + \frac{1}{2} \frac{\partial^2 V}{\partial \theta^2} = \frac{1}{2} \frac{\partial^2 V}{\partial \theta^2}.
\]

(44)

If \( G_0 \gg 1 \) and \( \sigma \geq 3 \) we can use the approximate solution of Eq. (44) which is known as the Fay’s solution

\[
V(\sigma, \theta) = \sum_{n=1}^{\infty} \frac{2}{G_0} \frac{\sin(n\theta)}{\sin \left( \frac{n(1 + \sigma)}{G_0} \right)}.
\]

(45)

If \((n(1 + \sigma))/G_0 < 1\) is small it is possible to use the following substitution in the Fay’s solution (45):

\[
\sin \left( \frac{n(1 + \sigma)}{G_0} \right) \approx \frac{n(1 + \sigma)}{G_0}.
\]

(46)

Substitution of the approximation (46) in the Fay’s solution yields

\[
V(\sigma, \theta) = \sum_{n=1}^{\infty} \frac{2}{n} \frac{\sin(n\theta)}{1 + \sigma},
\]

(47)

which is the approximate solution of the Burgers equation for ideal fluids.

Now, let us suppose the following approximate solution of the generalized Burgers equation Eq. (12):

\[
V(\sigma, \theta) = \text{Im} \left( \sum_{n=1}^{\infty} \frac{2}{G_0} \frac{A_n \exp(jn\theta)}{\sin \left( \frac{n(1 + \sigma A_n)}{G_0} \right)} \right).
\]

(48)

Provided that \( G_0 \gg 1 \) and \( D_0 \ll 1 \) then it is possible to use this simplification

\[
\frac{A_n}{\sin \left( \frac{n(1 + \sigma A_n)}{G_0} \right)} \approx \frac{A_n}{G_0}.
\]

(49)

If we use the approximation (49) in the solution (48), we obtain the solution (36).

At high frequencies the classical thermoviscous loss mechanism is dominant with respect to the wall losses (the classical attenuation \( \sim \omega^2 \), whereas the boundary layer attenuation \( \sim \sqrt{\omega} \)). In addition, at high frequency boundary layer dispersive effects are small enough and allow to approximate the phase speed \( c_{ph} \) by \( c_0 \). This follows from the relation
If we take into account the results mentioned above, we can assume that the solution (48) is the approximate solution of the generalized Burgers equation (12) in the region \( s \geq 3 \) when \( D_0 \ll 1 \).

### IV. COMPARISON OF APPROXIMATE SOLUTIONS

In this section we deal with comparison between the approximate (analytic) and numerical solutions of the generalized Burgers equation. The accuracy of the analytic solutions are investigated below for both the preshock and postshock region.

The Burgers equation (12) was solved by means of the standard Runge–Kutta method of the fourth order in the frequency domain (the first 100 harmonics were used). The numerical oscillations were damped by the method described by Fenlon\(^3\) in the postshock region. Each harmonic was multiplied by the coefficient \( \Psi_n \) given by

\[
\Psi_n = \frac{\sin(nH)}{nH}
\]

where \( H \) is the frequency damping coefficient. It causes the additional artificial attenuation of the solution. The value of \( H \) was chosen so that the numerical oscillations practically did not arise. No damping was used in the preshock region.

#### A. Preshock region

Spatial evolution curves for the first four harmonic amplitudes are shown in Figs. 1–3. To illustrate the accuracy of the analytic solution (30) we choose the Goldberg number \( G_0 \) that equals 50 and 500 and the value of parameter \( D_0 \), which represents the boundary layer effects, equal to 0.05 and 0.1. In order to demonstrate the advantage of the analytic solution we also depicted the Fubini solution in the figures. As shown in Figs. 1 and 2, the analytic predictions of the fundamental and second harmonic amplitudes are in good agreement with the numerical results. It is obvious that the accuracy of the Fubini solution is worse than in the case of the analytic one. As it is seen in these figures the accuracy of the analytic solution becomes worse for lower values of \( G_0 \) but it is better than the accuracy of the Fubini solution. In Fig. 3 spatial evolution curves for the harmonics are depicted...
when \( D_0 = 0.1 \). From comparison of the three solutions follows that the analytic prediction of the fundamental harmonic is in good agreement with the numerical solution but the Fubini solution failed.

The wave forms are shown in Figs. 4(a) and (b) for the different Goldberg number \((G_0 = 50 \text{ and } 500)\) and \( D_0 = 0.05 \) at the distance \( \sigma = 0.99 \). We can say on the basis of shown figures that the differences between the presented wave forms are very small, consequently we can consider sufficiently accurate both the analytic and Fubini solutions for the parameters \( G_0 \) and \( D_0 \) in the range which guarantees the validity of the analytic solution. We can see in Figs. 4(a) and (b) that the wave form asymmetry is not almost observable for small values of \( D_0 \).

### B. Postshock region

Both spatial evolution curves for the first four harmonic amplitudes and wave forms are depicted in the postshock region (behind the shock formation distance) in Figs. 5 and 6 and Figs. 7 and 8. The comparisons are made between the numerical solution, the Fay’s solution and the analytic one (48). As shown in the figures, the Fay’s solution disagrees with the numerical one when the boundary layer effects play a dominant role, that is in the case that the parameter \( D_0 \) is large relatively. In contrast with the Fay’s solution, the analytic solution is similar to the numerical one as we can see from the figures. In addition, the wave forms obtained from the analytic solution, which takes also into account the boundary layer effects, are asymmetric. The wave form asymmetry is caused by a boundary layer dispersion and occurs behind the shock formations because a phase shift between harmonics depends on a distance from the source. For the relatively large values of \( D_0 \) we can observe that there is a slight phase shift between the analytic and numerical solution, see Figs. 5 and 6. However, if values of \( D_0 \) are small then the agreement is excellent between both the solutions. As it is seen in the presented figures, the postshock analytic predictions are in good agreement with the numerical results even for higher values of \( D_0 \) contrary to the preshock analytic solution. Therefore the accuracy of the postshock analytic solution is better than the preshock analytic one.

There is a relatively good agreement between the Fubini solution and numerical one for small values of \( D_0 \), however the Fay’s solution apparently disagrees with the numerical one. Thus the advantage of the postshock analytic solution is evident in contrast to the Fay’s solution.

### V. CONCLUSION

This paper presents the derivation of the new approximate solution of the generalized Burgers equation in the preshock region \( 0 < \sigma < 1 \) for \( 1/G_0 \sim D_0 \sim \mu \). This approximate solution gives more precise results than the Fubini solution in the limits of its validity. The approximate solution converges to the Fubini solution when \( G_0 \to \infty, D_0 \to 0 \). The ap-
approximate solution is compared with the Fubini solution and the numerical one in the presented figures. Further, this paper contains the derivation of the new approximate solution of the generalized Burgers equation in the postshock region \( \sigma > 3 \) for \( D_0 \ll 1 \). This postshock approximate solution is contrasted with the Fay’s solution and the numerical one in the presented figures as well. The postshock approximate solution provides more precise results than the Fay’s solution in the limits of its validity because it takes into account the influence of the boundary layer which is not included in the Fay’s solution. The postshock approximate solution converges to the Fay’s solution for \( D_0 \to 0 \). Both the presented approximate solutions extend the existing area in which it is known the approximate solution of the generalized Burgers equation.

If we use for description of weakly nonlinear waves by the Khokhlov–Zabolotskaya–Kuznetsov (KZK) model equation in tubes, we can see that the boundary layer causes diffraction effects as well. From this reason it is necessary to limit using of the GBE with respect to the cutoff frequency. On the basis of the KZK numerical solution analysis it is apparent that the transversal velocity component grows significantly when the cutoff frequency is exceeded. No longer we can assume in this case that propagating waves remain plane or quasiplane and hence the GBE cannot be used. Thus if the plane wave with the frequency below the cutoff one propagates in the small radius tube, where the influence of the boundary layer is dominant, then a distribution of acoustic velocity along the radius varies only a little and the results of the KZK and GBE are comparable.

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