A versatile computational approach for the numerical modelling of parametric acoustic array

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(Received 17 May 2019; revised 31 July 2019; accepted 3 September 2019; published online 3 October 2019)

This work presents a versatile computational approach for the numerical modelling of a parametrically generated low-frequency sound. The proposed method is based on the quasi-linear approximation, and it does not employ the paraxial approximation. The primary acoustic field is calculated by the Rayleigh integral or the boundary element method; the secondary difference-frequency field is calculated by the finite element method. As governing wave equations, a general second-order wave equation for acoustic pressure, the Westervelt equation, and Kuznetsov equation are tested, and the corresponding numerical results are compared. The proposed approach allows studying the near-field, far-field, as well as the off-axis field of the difference-frequency wave parametrically radiated from complex emitters. As numerical examples, parametric radiation from a baffled piston and a piston combined with a horn are examined. © 2019 Acoustical Society of America.

https://doi.org/10.1121/1.5126863

I. INTRODUCTION

The concept of parametric acoustic array (PAA) was introduced in 1963 by Westervelt. This technique allows to generate low-frequency highly directional side-lobes-free sound beams from sources (radiators) with relatively small aperture. The principle is as follows; see, e.g., Refs. 1–3. Two collimated ultrasonic beams of similar frequencies are radiated in the same direction, where at least one of them has a finite amplitude. As these (primary) waves propagate in the medium, nonlinear effects give rise to the generation of a secondary field, one component of which is the (low-frequency) difference-frequency one, by forming a phased array of virtual sources that resonantly pump acoustic energy into this component. The directivity of this low-frequency wave is much higher than if it were radiated directly by the primary-field radiator.

Since its discovery the PAA has found its application in sonar, parametric loudspeakers, etc.; see, e.g., Refs. 4–9.

The process of the generation of a difference-frequency wave in a field of highly directional finite-amplitude primary waves is a rather complex one, and its mathematical description requires some degree of approximation and simplification. This problem has been addressed by many authors in different ways; some examples are given below.

In his pioneering work, Westervelt described the far-field properties of the difference-frequency waves based on the assumption that the nonlinear interactions are limited only to the near-field of the primary waves, which were modelled as collimated plane waves. Muir and Willette calculated the sum- and difference-frequency field under the assumption that the nonlinear interactions take place only in the far-field of the primary waves, which were modelled analytically using the formula for the far-field of a uniformly vibrating piston.

Garrett et al. proposed a model for the parametric radiation from an axisymmetric source in the quasilinear and parabolic approximation based on the numerical calculation of a triple integral. Kamakura et al. studied the propagation of high-amplitude waves generated by a piston vibrating at two similar frequencies by numerical integration of parabolic KZK equation. In Refs. 13–15, the method of multi-Gaussian beam expansion has been employed for the fast calculation of difference-frequency fields in the quasi-linear approximation, where in Refs. 13 and 14 the parabolic approximation has been employed for the calculation of the difference-frequency field. Nomura et al. simulated the parametric sound generation by means of direct numerical integration of Navier-Stokes equations in the time domain. This approach, in principle, allows to study more complex configurations; however, the numerical integration in the time domain over a large spatial domain (if the far-field behaviour is studied) results in a long computational time and a big amount of stored data, which need to be post-processed in order to obtain the spectral content of the sound field.

The aim of this work is to propose a versatile and computationally efficient approach to calculate the parametrically generated sound field from an arbitrary primary-wave source, not only a baffled planar piston. The only approximation adopted here is the quasi-linear one, which means that the acoustic field is assumed to be only weakly nonlinear.

The paper is organised as follows. In Sec. II, the second-order wave equations of nonlinear acoustics suitable for the calculation of the parametric radiation are reviewed, and the mutual relationships between them are mentioned. In Sec. III, the method of the successive approximations is employed to convert the model equations into a form suitable for the solution in the frequency domain. Section IV describes the numerical method used for the solution of the model equations. In Sec. V, predictions of the individual model equations are
compared in the case of a baffled planar piston radiator, and an example of a horned piston is given to demonstrate the versatility of the proposed approach. Section VI then draws some conclusions.

II. NONLINEAR WAVE EQUATIONS

The second-order nonlinear wave equation describing the wave propagation in a homogeneous and quiescent fluid reads \cite{2,17}

\[ \nabla^2 p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} + \frac{\delta}{c_0^2} \nabla^2 \frac{\partial p'}{\partial t} = -\frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 p'}{\partial t^2}, \tag{1} \]

where \( p' \) is the acoustic pressure, \( t \) is the time, \( c_0 \) is the small-signal adiabatic sound speed, \( \rho_0 \) is the ambient fluid density, \( \beta \) is the parameter of nonlinearity of the fluid for an ideal gas \( \beta = (\gamma + 1)/2 \), where \( \gamma \) is the adiabatic exponent, \( \delta = \frac{4\mu}{3} + \frac{\mu_B}{\gamma - 1} + \frac{1}{\gamma \rho_0} \) is the diffusivity of sound, where \( \mu, \mu_B \) are the coefficients of shear and bulk viscosity, respectively, \( \kappa \) is the coefficient of thermal conductivity, and \( c_p c_v \) are the specific heats at constant pressure and volume, respectively. The symbol \( \mathcal{L} \) stands for the Lagrangian density, which reads

\[ \mathcal{L} = \rho_0 \frac{v^2}{2} - \frac{p^2}{2\rho_0 c_0^2}, \tag{2} \]

where \( v^2 = \mathbf{v} \cdot \mathbf{v} \) is the square of the acoustic particle velocity vector. Equation (1) is an approximate wave equation valid to the second order in acoustic Mach number \( \varepsilon = v_0/c_0 \), where \( v_0 \) is the maximum acoustic particle velocity. Equation (1) accounts for mutual interactions of nonlinear and thermoviscous dissipative processes modifying the wave propagation in three-dimensional space.

In the second approximation, \( \mathcal{L} = 0 \) in the case of progressive plane waves. Then, the general model equation Eq. (1) reduces into the well-known Westervelt equation \cite{1,2,17}, which reads

\[ \nabla^2 p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} + \frac{\delta}{c_0^2} \nabla^2 \frac{\partial p'}{\partial t} = -\frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 p'}{\partial t^2}. \tag{3} \]

As it is discussed in Refs. 2 and 17, the Westervelt equation describes correctly (in the second-order approximation) plane progressive waves; however, even with the omission of the Lagrangian density, the cumulative nonlinear effects are still captured correctly even in the case of non-plane wave propagation, the Lagrangian density is associated only with local (non-cumulative) nonlinear effects. Due to its relative simplicity [compared to Eq. (1)], the Westervelt equation is widely used in the modelling of PAA.

If we introduce the velocity potential \( \varphi \) such that \( \mathbf{v} = \nabla \varphi \) and employ the second-order relationship between the acoustic pressure and the velocity potential; see, e.g., Ref. 18,

\[ p' = -\rho_0 \frac{\partial \varphi}{\partial t} - \rho_0 \frac{\partial}{2} (\nabla \varphi)^2 + \frac{\rho_0}{2\varepsilon_0} \left( \frac{\partial \varphi}{\partial t} \right)^2, \tag{4} \]

in the second approximation, wave equation Eq. (1) can be recast \cite{17} into

\[ \nabla^2 \varphi - \frac{1}{c_0^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{\delta}{c_0^2} \nabla^2 \frac{\partial \varphi}{\partial t} = \frac{1}{c_0^2} \left( \nabla \varphi \right)^2 + \frac{\beta - 1}{c_0^2} \left( \frac{\partial \varphi}{\partial t} \right)^2, \tag{5} \]

which is the well-known Kuznetsov equation \cite{19}. Equations (1) and (5) are derived under the same degree of approximation, and they are equivalent.

III. SUCCESSIVE APPROXIMATIONS

Within this work, acoustic fields are assumed to be only weakly nonlinear, which allows to employ the method of the successive approximations \cite{2} for solving the governing equations (1), (3), or (5). The quasi-linear solutions of the governing equations are assumed to have the form

\[ \psi(r, t) = \psi_1(r, t) + \psi_2(r, t), \tag{6} \]

where \( \psi \) represents the given acoustic variable (acoustic pressure, velocity potential), \( \psi_1 \) is the linear solution (approximation) of Eqs. (1), (3), or (5), representing the primary field, and \( \psi_2 \) is a small correction to \( \psi_1 \), representing the secondary field due to the nonlinear interactions. Obviously, \( |\psi_2| \ll |\psi_1| \).

The primary acoustic field is assumed to be composed of two time-harmonic fields, each at similar frequencies \( f_c \) (carrier) and \( f_s \) (side-band), where \( f_c > f_s \). Within this work, the difference-frequency component \( f_d \) of the secondary field is of interest, where \( f_d = f_c - f_s \). Being time-harmonic, all the components of the acoustic variables are further represented by their complex amplitudes \( \psi_j(r, t) = \Re[\psi_j(r) e^{i\omega_j t}] \), where \( j = c, s, d \), and \( i = \sqrt{-1} \).

For the primary-field components, the linearised equations (1), (3), or (5) reduce to the homogeneous Helmholtz equation

\[ \nabla^2 \psi_j + k_j^2 \psi_j = 0, \tag{7} \]

where \( j = c, s, \) and

\[ k_j^2 = \frac{\omega_j^2}{c_0^2}, \quad k_j^2 = \frac{\omega_j^2}{1 + i\omega_j \delta/c_0} \Rightarrow k_j \approx \frac{\omega_j}{c_0} - \frac{\omega_j^2 \delta}{2c_0^2}, \tag{8} \]

is the complex wavenumber, the imaginary part of which describes the thermoviscous attenuation. In audio- and low-ultrasonic-frequency regions, relaxation processes are responsible for most of the sound absorption, which in air is dependent on temperature, atmospheric pressure, and water vapor content. The sound attenuation coefficient for the given frequency \( \omega_j \) can be generalized \cite{20,21} to account for these effects.

For the difference-frequency secondary field, Eqs. (1) and (3) reduce to the inhomogeneous Helmholtz equation
\[ \mathbf{V}^2 \tilde{p}_d + k_d^2 \tilde{p}_d = \tilde{q}_{pd}, \]  
\[ \text{where the source term} \]
\[ \tilde{q}_{pd} = \frac{\beta_0 \omega^2}{\rho_0 c_0^2} \tilde{p}_e \tilde{p}_c^* - \left( \mathbf{V}^2 - k_d^2 \right) \tilde{L}_d, \]  
\[ \text{where} \]
\[ \tilde{L}_d = \frac{\rho_0}{2} \tilde{v}_e \cdot \tilde{v}_s^* - \frac{\rho_0}{2 \rho_0 c_0} \tilde{p}_e \tilde{p}_c^* \]  
\[ \text{for Eq. (1). The asterisk stands for the complex conjugate,} \]
\[ \tilde{L}_d = 0 \]  
\[ \text{for the Westervelt equation [Eq. (3)]}. \]
\[ \text{For the difference-frequency secondary field, Eq. (5) reduces to} \]
\[ \mathbf{V}^2 \tilde{\phi}_d + k_d^2 \tilde{\phi}_d = \tilde{q}_{od}, \]  
\[ \text{where} \]
\[ \tilde{q}_{od} = \frac{i \omega_d}{c_0^2} \left[ \tilde{v}_e \cdot \tilde{v}_s^* + (\beta - 1) \frac{\omega_e \omega_c}{c_0} \tilde{\varphi}_e \tilde{\varphi}_c^* \right]. \]  
\[ \text{Using the difference-frequency, and primary-field velocity potential, the difference-frequency acoustic pressure can be calculated, employing Eq. (4), as} \]
\[ \tilde{p}_d = -i \omega_d \rho_0 \tilde{\varphi}_d - \frac{\rho_0}{2} \tilde{v}_e \cdot \tilde{v}_s^* + \frac{\rho_0 \omega_e \omega_c}{2 c_0^2} \tilde{\varphi}_e \tilde{\varphi}_c^*. \]

**IV. NUMERICAL METHOD**

As it has been shown above, employing the method of the successive approximations for the time-harmonic fields, the governing nonlinear wave equations (1), (3), or (5) reduce to the homogeneous Helmholtz equation (7) for the primary field, and into the inhomogeneous Helmholtz equations (9) and (12) for the difference-frequency secondary field. The source terms [Eqs. (10) and (13)] for the secondary-field equations are calculated from the primary-field quantities.

The original idea was to calculate the primary-field, as well as the secondary-field quantities employing the finite element method (FEM). However, as the general rule of thumb states that there are at least six elements per wavelength needed for the discretization, calculation of the high-frequency primary field in a large computational domain (external radiation problem) resulted in an extreme amount of the computer-memory needed. Moreover, in the case of Eq. (1), where there are second spatial derivatives of the primary-field quantities needed for the calculation of the source terms [Eqs. (10) and (11)], the computational mesh must be way more fine.

In order to circumvent these obstacles, the following two approaches have been tested for the calculation of the high-frequency primary field quantities: (a) for the case of the radiation from a baffled piston, the Rayleigh integral was evaluated numerically; (b) the boundary element method (BEM) was employed. In both the cases, the low-frequency secondary field was calculated employing FEM in COMSOL Multiphysics (Acoustic Module, Pressure Acoustics Interface, frequency-domain, COMSOL AB, Stockholm, Sweden).

**A. Rayleigh integral**

Geometrical arrangement of the piston baffled in an infinite rigid plane is shown in Fig. 1.

The complex amplitude of the acoustic pressure of the primary field at point \( P \), for the calculation of the source terms [Eqs. (10) and (13)], can be calculated employing the Rayleigh integral (see, e.g., Ref. 22) as

\[ \tilde{p}_j(r) = \frac{i k_j \rho_0 c_0}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}_j(x',y') e^{-ik_j r} \frac{\epsilon}{R} \, dx' \, dy', \]
\[ j = c, s, \]
\[ R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}. \]

The complex amplitude of the velocity potential is then calculated using the formula \( \tilde{\varphi}_j = ip_j / k_j \rho_0 c_0 \), and from here, the \( x \)- and \( z \)-components of the acoustic particle velocity vector

\[ \tilde{v}_j(x) = \frac{\partial \tilde{\varphi}_j}{\partial x} = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}_j(x',y') (x-x') \]
\[ \times (1 + ik_j R) e^{-ik_j R} \frac{\epsilon}{R^3} \, dx' \, dy', \]
\[ \tilde{v}_z(r) = \frac{\partial \tilde{\varphi}_j}{\partial z} = \frac{z}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}_j(x',y') (1 + ik_j R) \]
\[ \times \frac{e^{-ik_j R}}{R^3} \, dx' \, dy'. \]

The \( x \)-component of the acoustic particle velocity vector in Eq. (16a) represents a component perpendicular to the \( z \)-direction.
B. BEM

Within the BEM, the boundary conditions are used to fit the boundary values on the surfaces. Subsequently, the acoustic field quantities in the fluid domain are calculated using the boundary values during the post-processing. This technique is usually more efficient (than, e.g., FEM) in the cases where the surface/volume ratio of the computational domain is small, which is the case of the problem studied within this work (external radiation problem). Compared to the Rayleigh integral (see Sec. IV A), radiation from more complex sources can be studied.

Within this work, the high-frequency primary-field acoustic quantities and the source terms [Eqs. (10) and (13)] were calculated using Pressure Acoustics, Boundary Elements Interface of Acoustics Module of COMSOL Multiphysics, exported as text files, and imported to the Acoustic Module, Pressure Acoustics Interface of COMSOL Multiphysics to calculate the secondary field by FEM.

V. NUMERICAL RESULTS

Within all the examples shown in this section, acoustic field is calculated in air at normal conditions, and classical thermoviscous attenuation is taken into account. Namely, \( c_0 = 343.2 \text{ m s}^{-1} \), \( \rho_0 = 1.204 \text{ kg m}^{-3} \), \( \gamma = 1.4 \), and \( \delta = 3.764 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \).

In all the following cases, the carrier frequency \( f_c = 40 \text{ kHz} \). The primary-field source (radiator) is assumed to have an axial symmetry so that the acoustic field is calculated in \((r,z)\) axisymmetric cylindrical coordinates.

A. Baffled piston

Here, radiation from a baffled circular piston (see Fig. 1) of radius \( a = 2 \text{ cm} \) is examined. The complex amplitude of its vibration velocity at frequencies \( f_c, f_d \) is assumed to have the form

\[
\tilde{u}_r(r) = u_0 \quad \text{(uniform distribution),}
\]

\[
\tilde{u}_r(r) = u_0 (1 - r^2/a^2) \quad \text{(parabolic distribution),}
\]

\[
\tilde{u}_r(r) = u_0 (1 - r^2/a^2)^2 \quad \text{(quartic distribution),}
\]

(see Fig. 2) where \( f = c, s, \) \( r = \sqrt{x^2 + y^2} \leq a \) and, for simplicity, \( u_0 = 1 \text{ m s}^{-1} \). In all the numerical results presented in this subsection, the primary acoustic field is calculated employing the Rayleigh integral (see Sec. IV A), however, BEM provides the same results.

Figure 3 shows the distribution of the amplitude of the acoustic pressure of the difference-frequency wave \((f_d = 1 \text{ kHz})\) along the \( z \) axis in the near-field of the carrier wave (for \( f_c = 40 \text{ kHz} \), the Rayleigh distance \( R_r = \frac{\pi k_c a^2}{2} = 14.6 \text{ cm} \)). The velocity-distribution of the piston is the parabolic one; see Eq. (17).

The difference-frequency field is calculated employing Eq. (1) (red line in Fig. 3), the Westervelt equation [Eq. (3), blue line in Fig. 3], and the Kuznetsov equation [Eq. (5), green line in Fig. 3]. As Eq. (1) and the Kuznetsov equation are derived under the same approximation, the numerical results correspond to each other very well. As in the Westervelt equation, the Lagrangian density [Eq. (2)] is assumed to be zero, which is not fulfilled in the near-field of the primary wave, and the prediction by the Westervelt equation differs from Eq. (1) and the Kuznetsov equation. However, in the far-field, beyond the Rayleigh distance, the predictions of all the three equations match each other as is expected.

As the numerical evaluation of the source term [Eq. (10)] for Eq. (1) is much more expensive than the source term [Eq. (13)] for the Kuznetsov equation, and as it is necessary to calculate the second-order spatial derivatives of the primary-field quantities, the Kuznetsov equation [Eq. (5)] is employed for the calculation of the near-field further on.

Figure 4 shows the acoustic pressure amplitude of the difference-frequency wave along the \( z \) axis for \( f_d = 1 \text{ kHz} \) and individual piston velocity distributions [see Eq. (17)]. The solid lines correspond to the predictions by the Kuznetsov equation, and the dashed lines are related to the Westervelt equation. It can be observed that the structure of the near-field (resolved by the Kuznetsov equation) depends on the piston velocity distribution and, again, the predictions of both the equations correspond to each other in the far-field.
The difference in the amplitudes (at a given point in the far-field) for the individual piston velocity distributions are caused by the fact that the amount of radiated acoustic energy for the individual cases differs.

Figure 5 shows the normalized acoustic pressure amplitude of the difference-frequency wave along the z-axis for the parabolic piston velocity distribution and individual frequencies $f_d$. The waveforms for the individual frequencies are normalized by the factor $(f_d/1\, \text{kHz})^2$ for better legibility of the details. The solid lines correspond to the predictions by the Kuznetsov equation, and the dashed lines are associated with the Westervelt equation. It can be seen that the near-field structure resolved by the Kuznetsov equation depends on the difference-wave frequency and is more prominent for lower frequencies $f_d$. In all the cases in the far-field, the predictions by both the equations match.

Figure 6 shows the directivity function $D(\varphi) = \left| \hat{p}(r_0, \varphi) / \hat{p}(r_0, 0) \right|$ calculated at the distance $r_0 = 2\, \text{m}$ from the centre of the piston with the parabolic velocity distribution. It can be observed that the low-frequency difference-frequency field (solid lines) has a similar directivity as the high-frequency carrier wave (dashed line), and the directivity of the difference-frequency wave increases with its frequency. Radiation from this piston directly at the frequency shown in Fig. 6 (500 Hz–5 kHz) would be omnidirectional.

Figure 7 shows the directivity function calculated at the distance $r_0 = 2\, \text{m}$ from the centre of the piston with various velocity distributions [Eq. (17)] for the difference-frequency wave at $f_d = 1\, \text{kHz}$. It can be observed that the uniform velocity distribution provides the highest directivity, which is associated with the fact that, in this case, the entire piston surface vibrates with the same velocity, compared to the case of the parabolic or quartic distributions, where the vibration velocity decreases toward the piston edge.

B. Horned piston

Within this subsection, the capabilities of the proposed computational approach are demonstrated on an example of a piston equipped with a simple horn. It has been demonstrated in the experimental work\textsuperscript{23} that if a parametric radiator is equipped with a horn, its directivity and efficiency strongly increases; later, a simple model\textsuperscript{24} is proposed.

In this work, the horn is supposed to have a hyperbolic-cosine shape, described as

$$r(z) = a \cosh(a z), \quad \text{where} \quad z = \frac{1}{l} \text{arcosh} \left( \frac{b}{a} \right), \quad (18)$$

(see also Fig. 8) where $a$ is the small horn radius (throat, the same as the piston radius), $b$ is the larger horn radius (mouth), and $l$ is the horn length.

In all the numerical results presented in this subsection, $a = 2\, \text{cm}$, $b = 4\, \text{cm}$, and $f_c = 40\, \text{kHz}$; the primary acoustic field is calculated employing BEM (see Sec. IV B).

Figure 9 shows the distribution of the normalized acoustic pressure amplitude of the carrier wave along the z-axis for uniform piston velocity distribution and individual lengths $l$ of the horn. It can be observed that the structure of the near-field is very much dependent on the horn length; however, the far-field is not affected by the presence of the horn significantly. The far-field directivity is similar for all the cases, and with the increasing horn length, the on-axis pressure slightly increases. For example, at the distance of

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FIG. 4. (Color online) Difference-frequency wave along the z axis for $f_d = 1\, \text{kHz}$, and individual piston velocity distributions. Predictions are shown by the Kuznetsov equation (solid lines) and Westervelt equation (dashed lines).

FIG. 5. (Color online) Normalized difference-frequency wave along the z axis for the parabolic piston velocity distribution and individual difference frequencies. Predictions are shown by the Kuznetsov equation (solid lines) and Westervelt equation (dashed lines).

FIG. 6. (Color online) Directivity function of the piston with the parabolic velocity distribution calculated at the distance $r_0 = 2\, \text{m}$ from the piston centre; dashed line, the primary wave (40 kHz); solid lines, individual difference frequencies.

FIG. 7. (Color online) Directivity function of a piston with various velocity distributions calculated at the distance $r_0 = 2\, \text{m}$ from the piston centre, $f_d = 1\, \text{kHz}$. 
By about 11% higher than without the horn. The primary wave radiation efficiency is not influenced by the presence of the horn significantly because in this case \( R(k_c/a) = 14.6 \gg 1 \), which means that the piston radiation impedance is close to the characteristic one (the presence of the horn does not influence it significantly), and the beam emitted by the piston alone is rather directional—the first null-radiation angle for the carrier wave is \( \theta_1 = 15^{\circ} 10' \).

Figure 10 shows the acoustic pressure amplitude of the difference-frequency wave at \( f_d = 1 \) kHz along the z axis for the same conditions as before. The solid lines represent the predictions by the Kuznetsov equation, and the dashed lines are related to the Westervelt equation. It can be observed that with the increasing horn length, the acoustic pressure in the vicinity of the piston rapidly increases, and this region of the increased pressure is more-or-less limited to the dimensions of the horn. Farther away from the horn (in the far-field), the acoustic pressure amplitude is not very much dependent on the horn length. In contrast to the predictions by the Westervelt equation, the Kuznetsov equation predicts small ripples in the near-field, probably caused by the existence of a partial standing wave in the horn.

The influence of the presence of a horn on the difference-frequency wave in the far-field is demonstrated in Fig. 11. All the parameters are the same as before, and the acoustic field is calculated at the distance \( r_0 = 2 \) m from the piston centre. It can be seen that with the increasing horn length (from \( l = 0 \) cm to \( l = 6 \) cm), the on-axis pressure increases with the horn length. For example, for the horn with length \( l = 6 \) cm, the on-axis pressure is 11.3% higher than without the horn. But, with the horn length increased to

\( l = 8 \) cm, the on-axis pressure decreases, and its amplitude is similar to the case of \( l = 4 \) cm. Also, the presence of a “side-lobe” can be observed in this case.

Figure 12 shows the directivity pattern of the difference-frequency wave for the same parameters as shown in Fig. 11 but for the piston with parabolic velocity distribution. Contrary to the previous case with increasing horn length \( l \), the on-axis pressure amplitude decreases. As before, the presence of a side-lobes can be observed in the case of \( l = 8 \) cm.

VI. CONCLUSIONS

Within this work, a versatile computational approach for the numerical modelling of parametric generation of a low-frequency sound has been described. The proposed method is based on the quasi-linear approximation and it does not employ the paraxial approximation. The primary acoustic field is calculated by BEM or employing the Rayleigh integral in the case of a baffled piston; the secondary difference-frequency field is calculated by FEM. Three nonlinear wave equations derived in the second approximation have been compared: a general equation for the acoustic pressure, the Westervelt equation, and the Kuznetsov equation for the velocity potential. It has been demonstrated that as the Lagrangian density is neglected in the Westervelt equation, it does not correctly capture the near-field complexity of the secondary difference-frequency field. All the equations provide the same results in the far-field. If the information about the near-field structure of the difference-frequency wave is needed, the Kuznetsov equation for the velocity potential is the most suitable model equation as its nonlinear terms do not contain the second-order spatial

FIG. 9. (Color online) Normalized acoustic pressure amplitude of the carrier wave at \( f_c = 40 \) kHz along the z axis for the uniform piston velocity distribution and individual horn lengths.
derivatives whose numerical evaluation is expensive and prone to the numerical discretization errors. Two numerical examples have been given. First, the near-field, as well as the far-field, of a difference-frequency wave parametrically radiated from a baffled circular piston with different velocity distributions have been examined. Second, the radiation pattern of a parametric emitter combined with a horn has been studied. The numerical results show that in the studied cases, the directivity of the difference-frequency wave can be increased as well as decreased by the presence of the horn, depending on its geometry and the piston velocity distribution. These findings are in accord with the results reported in the experimental work, where, unfortunately, the geometrical details have not been given. Contrary to the experimental work—a prominent directivity improvement, thanks to the horn, has not been observed, which means that this effect may be rather delicate, and it requires further study.

ACKNOWLEDGMENTS

This work was supported by Grant Agency of the Czech Republic (GACR) Grant No. GA18-24954S.